

BCA 3rd Year

Numerical Methods

UNIT 1

Roots of Equations

* Numerical Methods — The limitation of analytical method in practical applications have lead scientific and engineers to evaluate numerical methods with the help of CBOT (Computer Based Optimization Techniques) (Numerical analysis), we can solve large problem with the accuracy result.

The aim of numerical analysis is therefore to provide constructive methods for obtaining answers to such problems in a numerical form.

* Accuracy of numbers :-

1- Approximate numbers — There are two types of numbers

(i) Exact numbers like — 2, 4, 9, $\frac{7}{2}$

Approximate numbers like — 3.1444, 2.788

2- Round of / Significant numbers — Significant numbers like — 3456, 3.4546.

* Bisection Method :-

Ques 1: Find a root of the equation $x^3 - 4x - 9 = 0$ using Bisection method in four stages.

Sol:-

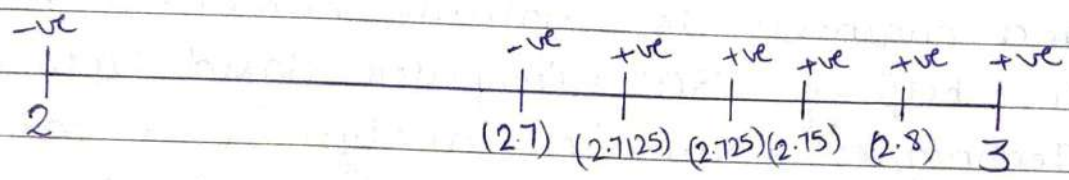
$$f(x) = x^3 - 4x - 9$$

$$f(1) = 1^3 - 4 - 9 = -12 \text{ (-ve)}$$

$$f(2) = 2^3 - 8 - 9 = -9 \text{ (-ve)}$$

$$f(3) = 3^3 - 12 - 9 = 6 \text{ (+ve)}$$

The root lies between 2 and 3. We find nearest root with the help of points 2 and 3 then we find the value.



$$f(2.5) = (2.5)^3 - 4 \times 2.5 - 9 = -3.375 \text{ (-ve)}$$

$$f(2.6) = (2.6)^3 - 4 \times 2.6 - 9 = -1.824 \text{ (-ve)}$$

$$f(2.7) = (2.7)^3 - 4 \times 2.7 - 9 = -0.117 \text{ (-ve)}$$

$$f(2.8) = (2.8)^3 - 4 \times 2.8 - 9 = 1.752 \text{ (+ve)}$$

We find first approximation to the root is -

$$x_1 = \frac{2.7 + 2.8}{2} = 2.75$$

$$f(2.75) = (2.75)^3 - 4 \times 2.75 - 9 = 0.7968 \text{ (+ve)}$$

The root lies between 2.7 and 2.75. We find second approximation to the root is

$$x_2 = \frac{2.7 + 2.75}{2} = 2.725$$

$$f(2.725) = (2.725)^3 - 4 \times 2.725 - 9 = 0.3348 \text{ (+ve)}$$

The root lies between 2.7 and 2.725. We find third approximation to the root is

$$x_3 = \frac{2.7 + 2.725}{2} = 2.7125$$

$$f(2.7125) = (2.7125)^3 - 4 \times 2.7125 - 9 = 0.1076 \text{ (+ve)}$$

The root lies between 2.7 and 2.7125. We find fourth approximation to the root is

$$x_4 = \frac{2.7 + 2.7125}{2} = \underline{2.70625}, \text{ Ans.}$$

$$f(2.706) = (2.706)^3 - 4 \times 2.706 - 9 = -5 \text{ (-ve)}$$

Ques 2-

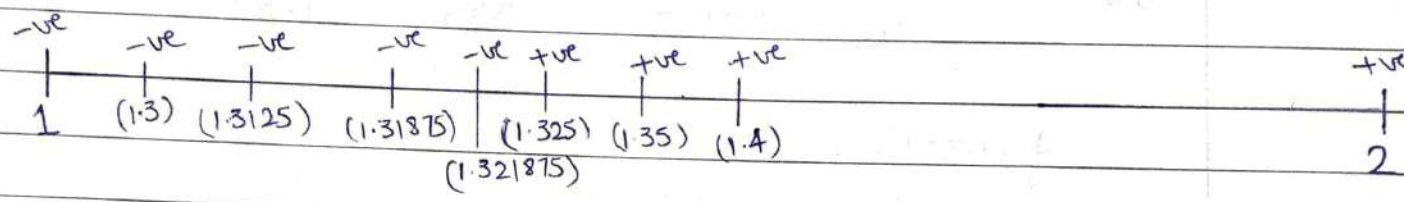
$$x^3 - x - 1 = 0 \quad (\text{five stages})$$

$$f(x) = x^3 - x - 1$$

$$f(1) = 1^3 - 1 - 1 = -1 \text{ (-ve)}$$

$$f(2) = 2^3 - 2 - 1 = 5 \text{ (+ve)}$$

The root lies between 1 and 2.



$$f(1.2) = (1.2)^3 - 1.2 - 1 = -0.472 \text{ (-ve)}$$

$$f(1.3) = (1.3)^3 - 1.3 - 1 = -0.103 \text{ (-ve)}$$

$$f(1.4) = (1.4)^3 - 1.4 - 1 = 0.344 \text{ (+ve)}$$

We find first approximation to the root is -

$$x_1 = \frac{1.3 + 1.4}{2} = 1.35$$

$$f(1.35) = (1.35)^3 - 1.35 - 1 = 0.110 \text{ (+ve)}$$

The root lies between 1.3 and 1.35. We find second approximation to the root is -

$$x_2 = \frac{1.3 + 1.35}{2} = 1.325$$

$$f(1.325) = (1.325)^3 - 1.325 - 1 = 1.203 \text{ (+ve)}$$

The root lies between 1.3 and 1.325. We find third approximation to the root is -

$$f(1.3125) = (1.3125)^3 - 1.3125 - 1 = -0.0515 \text{ (-ve)}$$

The root lies between 1.3125 and 1.325. We find fourth approximation to the root is -

$$x_4 = \frac{1.3125 + 1.325}{2} = 1.31875$$

$$f(1.31875) = (1.31875)^3 - 1.31875 - 1 = -0.0253 \text{ (-ve)}$$

The root lies between 1.31875 and 1.325. We find fifth approximation to the root is

$$x_5 = \frac{1.31875 + 1.325}{2} = 1.321875$$

$$f(1.321) = (1.321)^3 - 1.321 - 1 = -0.015 \text{ (-ve)}$$

Ques 3-

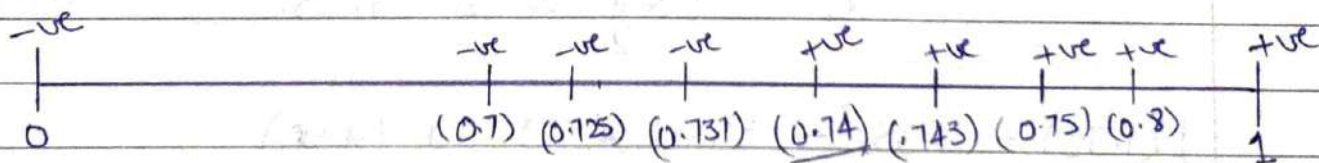
$$x - \cos x = 0$$

$$f(x) = x - \cos x$$

$$f(0) = 0 - \cos 0 = -1 \text{ (-ve)}$$

$$f(1) = 1 - \cos 1 = 1 \text{ (+ve)}$$

The root lies between 0 and 1.



$$f(0.5) = 0.5 - \cos 0.5 = -0.377 \text{ (-ve)}$$

$$f(0.6) = 0.6 - \cos 0.6 = -0.225 \text{ (-ve)}$$

$$f(0.7) = 0.7 - \cos 0.7 = -0.064 \text{ (-ve)}$$

$$f(0.8) = 0.8 - \cos 0.8 = 0.103 \text{ (+ve)}$$

First approximation $x_1 = \frac{0.7 + 0.8}{2} = 0.75$

Second approximation -

$$x_2 = \frac{0.7 + 0.75}{2} = 0.725$$

$$f(0.725) = 0.725 - \cos 0.725 = -0.023 (-ve)$$

Third approximation -

$$x_3 = \frac{0.725 + 0.75}{2} = 0.7375$$

$$f(0.7375) = 0.7375 - \cos 0.7375 = -3.488 (-ve)$$

Fourth approximation -

$$x_4 = \frac{0.7375 + 0.75}{2} = 0.74375$$

$$f(0.743) = 0.743 - \cos 0.743 = 6.557 (+ve)$$

Fifth approximation -

$$x_5 = \frac{0.7375 + 0.743}{2} = 0.74$$

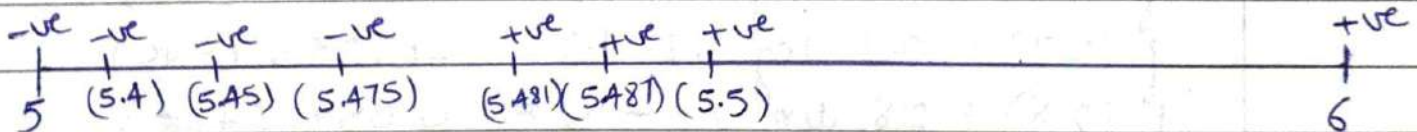
$$f(0.74) = 0.74 - \cos 0.74 = 1.53 (+ve)$$

ues 4

$$f(x) = x^2 - 30$$

$$f(5) = (5)^2 - 30 = -5 (-ve)$$

$$f(6) = (6)^2 - 30 = 6 (+ve)$$



$$f(5.3) = (5.3)^2 - 30 = -1.91 (-ve)$$

$$f(5.4) = (5.4)^2 - 30 = -0.84 (-ve)$$

$$f(5.5) = (5.5)^2 - 30 = 0.25 (+ve)$$

First approximation -

$$x_1 = \frac{5.4 + 5.5}{2} = 5.45$$

$$f(5.45) = (5.45)^2 - 30 = -0.297 (-ve)$$

Second approximation -

$$x_2 = \frac{5.45 + 5.5}{2} = 5.475$$

$$f(4.7) = f(5.475) = (5.475)^2 - 30 = -0.024 \text{ (-ve)}$$

Third approximation -

$$x_3 = \frac{5.475 + 5.5}{2} = 5.4875$$

$$f(5.4875) = (5.4875)^2 - 30 = 0.112 \text{ (+ve)}$$

Fourth approximation -

$$x_4 = \frac{5.475 + 5.4875}{2} = \boxed{5.48125}$$

$$f(5.481) = (5.481)^2 - 30 = 0.041 \text{ (+ve)}$$

Ques 5-

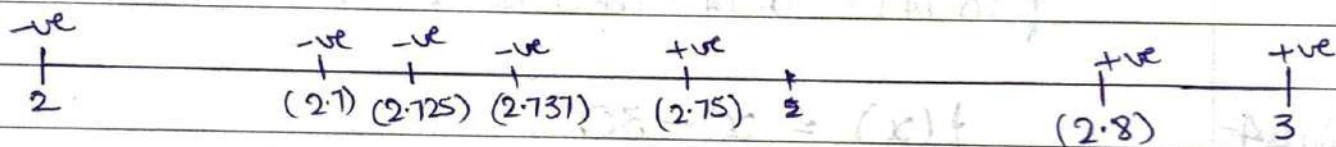
$$x \log x - 1.2 = 0$$

$$f(x) = x \log x - 1.2$$

$$f(1) = 1 \log 1 - 1.2 = -1.2 \text{ (-ve)}$$

$$f(2) = 2 \log 2 - 1.2 = -0.59 \text{ (-ve)}$$

$$f(3) = 3 \log 3 - 1.2 = 0.23 \text{ (+ve)}$$



$$f(2.5) = 2.5 \log 2.5 - 1.2 = -0.205 \text{ (-ve)}$$

$$f(2.6) = 2.6 \log 2.6 - 1.2 = -0.121 \text{ (-ve)}$$

$$f(2.7) = 2.7 \log 2.7 - 1.2 = -0.035 \text{ (-ve)}$$

$$f(2.8) = 2.8 \log 2.8 - 1.2 = 0.052 \text{ (+ve)}$$

First approximation -

$$x_1 = \frac{2.7 + 2.8}{2} = 2.75$$

$$f(2.75) = 2.75 \log 2.75 - 1.2 = 0.08 \text{ (+ve)}$$

Second approximation -

$$x_2 = \frac{2.7 + 2.75}{2} = 2.725$$

$$f(2.725) = 2.725 \log 2.725 - 1.2 = -0.01 \text{ (-ve)}$$

$$x_3 = \frac{2.125 + 2.75}{2} = 2.7375$$

$$f(2.7375) = 2.737 \log 2.737 - 1.2 = -0.003 \text{ (-ve)}$$

Fourth approximation -

$$x_4 = \frac{2.737 + 2.75}{2} = 2.7435$$

$$f(2.743) = 2.743 \log 2.743 - 1.2 = 2.7435 \text{ (+ve)}$$

* False Position Method :-

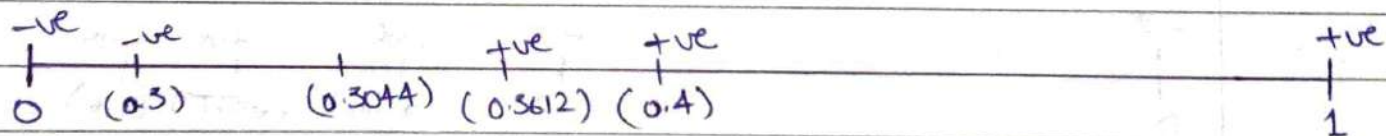
Quest- Find a root of the eqⁿ $3x + \sin x - e^x = 0$ by the method of false position correct to four decimal places.

Sol:-

$$f(x) = 3x + \sin x - e^x$$

$$f(0) = 0 + \sin 0 - e^0 = -1 \text{ (-ve)}$$

$$f(1) = 3 + \sin 1 - e^1 = 2.84 \text{ (+ve)}$$



$$f(0.5) = 3 \times 0.5 + \sin 0.5 - e^{0.5} = 0.330 \text{ (+ve)}$$

$$f(0.6) = 3 \times 0.6 + \sin 0.6 - e^{0.6} = 0.542 \text{ (+ve)}$$

$$f(0.4) = 3 \times 0.4 + \sin 0.4 - e^{0.4} = 0.0975 \text{ (+ve)}$$

$$f(0.3) = 3 \times 0.3 + \sin 0.3 - e^{0.3} = -0.154 \text{ (-ve)}$$

where $x_0(0.3)$ is negative and $x_1(0.4)$ is positive then root lies between 0.3 and 0.4.

Using Regular false method -

$$x_2 = x_0 - \frac{x_1 - x_0 \cdot f(x_0)}{f(x_1) - f(x_0)}$$

$$= 0.3 + \frac{0.4 - 0.3}{0.0975 + 0.154} (0.154)$$

$$x_2 = 0.3612$$

$$f(0.3612) = 3 \times 0.3612 + \sin(0.3612) - e^{0.3612} = 1.946 (+ve)$$

The root lies between $x_1 = 0.3$ and $x_0 = 0.3612$

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0.3612 - \left(\frac{0.3 - 0.3612}{-0.154 - 1.946} \right) (1.946)$$

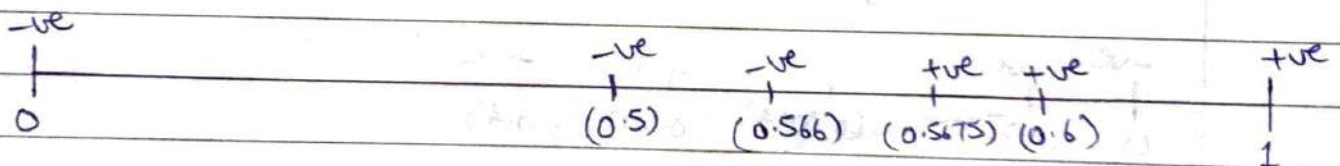
$$x_3 = 0.3044$$

Ques 2. $x - e^x = 0$ Solve by Regular Falsi method.

Let $f(x) = x - e^x$

$$f(0) = 0 - e^0 = -1 (-ve)$$

$$f(1) = 1 - e^1 = 0.632 (+ve)$$



$$f(0.5) = 0.5 - e^{-0.5} = -0.106 (-ve)$$

$$f(0.6) = 0.6 - e^{-0.6} = 0.051 (+ve)$$

The root lies between 0.5 and 0.6 where $x_0 = 0.5$, $x_1 = 0.6$

Using regular falsi method we have

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$x_2 = 0.5 - \left(\frac{0.6 - 0.5}{0.051 + 0.106} \right) (-0.106)$$

$$= 0.5 + 0.067$$

$$x_2 = 0.5675$$

$$f(0.5675) = 0.5675 - e^{-0.5675} = 0.00055 \text{ (+ve)}$$

The root lies between x_0 (0.567) and x_1 (0.567)

$$x_3 = 0.5 - \left(\frac{0.567 - 0.5}{0.00055 + 0.106} \right) (-0.106)$$

$$= 0.5 + 0.066$$

$$x_3 = 0.566$$

$$f(0.566) = 0.566 - e^{-0.566} = -0.0017 \text{ (-ve)}$$

Two values are matched to decimal two places. The answer is 0.566.

Ques 3-

$$x^3 - 5x + 3 = 0$$

$$f(x) = x^3 - 5x + 3$$

$$f(0) = 0 - 0 + 3 = 3 \text{ (+ve)}$$

$$f(1) = (1)^3 - 5 \times 1 + 3 = -1 \text{ (-ve)}$$

The root lies between 0 and 1.

+ve
|
0

+ve +ve +ve +ve
| | | |
(0.5) (0.6) (0.6294) (0.6410)

-ve -ve
| |
(0.7) 1

$$f(0.5) = (0.5)^3 - 5 \times 0.5 + 3 = 0.625 \text{ (+ve)}$$

$$f(0.6) = (0.6)^3 - 5 \times 0.6 + 3 = 0.216 \text{ (+ve)}$$

$$f(0.7) = (0.7)^3 - 5 \times 0.7 + 3 = -0.517 \text{ (-ve)}$$

The root lies between 0.6 and 0.7 where $x_0 = 0.6$, $x_1 = 0.7$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0.6 - \left(\frac{0.7 - 0.6}{-0.517 - 0.216} \right) (0.216)$$

$$= 0.6 + 0.0294$$

$$x_2 = 0.6294$$

$$f(0.6294) = (0.6294)^3 - 5 \times 0.6294 + 3 = 0.1023 \text{ (+ve)}$$

The root lies between 0.6294 and 0.7 where $x_0 = 0.6294, x_1 = 0.7$

$$x_3 = 0.6294 - \left(\frac{0.7 - 0.6294}{-0.517 - 0.1023} \right) (0.1023)$$

$$= 0.6294 + 0.0116$$

$$x_3 = 0.6410$$

$$f(0.6410) = (0.6410)^3 - 5 \times 0.6410 + 3 = 0.0583 \text{ (ve)}$$

The root lies between 0.6410 and 0.7

$$x_4 = 0.6410 - \left(\frac{0.7 - 0.6410}{-0.517 - 0.0583} \right) (0.0583)$$

$$= 0.6410 + 0.00597$$

$$= 0.6469$$

Two values are matched to decimal two points. The answer is 0.6469.

* Newton-Raphson Method (Newton iterative method)

Q1- Find the real root of the eqⁿ $x \log_{10} x = 1.2$ correct to five decimal places.

$$f(x) = x \log_{10} x - 1.2$$

$$f(1) = -1.2$$

$$f(2) = -0.597 \text{ (-ve)}$$

$$f(3) = 0.2313 \text{ (+ve)}$$

The root lies between 2 and 3

$$f(2.5) = -0.205$$

$$f(2.7) = -0.0353 \text{ (-ve)}$$

$$f(2.8) = 0.0520 \text{ (+ve)}$$

The root lies between 2.7 and 2.8.

$$f'(x) = x \cdot \log_e \cdot \frac{1}{x} + \log x \cdot x$$

$$f'(x) = \log_e + \log x$$

$$f'(x) = 0.43429 + \log_{10} x$$

Let taking $x = 2.7$

By Newton iterative method -

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left[\frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.43429} \right]$$

$$= \frac{x_n \log_{10} x_n + 0.43429 x_n - x_n \log_{10} x_n + 1.2}{\log_{10} x_n + 0.43429}$$

$$x_{n+1} = \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429}$$

First iteration, put $n=0$ then

$$x_1 = \frac{0.43429 x_0 + 1.2}{\log_{10} x_0 + 0.43429} \quad [\text{where } x_0 = 2.7]$$

$$= \frac{0.43429(2.7) + 1.2}{\log_{10}(2.7) + 0.43429} = 2.74079$$

Second iteration, we have $n=1$, $x_1 = 2.74079$

$$x_2 = \frac{0.43429(2.74079) + 1.2}{\log_{10}(2.74079) + 0.43429}$$
$$= 2.74064$$

Third iteration,

$$x_3 = \frac{0.43429(2.74064) + 1.2}{\log_{10}(2.74064) + 0.43429}$$
$$= 2.74064$$

The value has matched upto 5 decimal points.

The correct root is ~~2.70~~ 2.74064.

Interpolation and Extrapolation

★

Finite difference formula

Forward

Backward

Q1- Find difference forward formula-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962	-114	-75	2	-1
10	9848	-189	-73	1	2
15	9659	-262	-72	3	
20	9397	-334	-69		
25	9063	-403			
30	8660				

Find the value of $\Delta^2 y_{10}$, $\Delta^4 y_5$

$$\Delta^2 y_{10} = -73$$

$$\Delta^4 y_5 = -1$$

Ques 2- If $y = x^3 + x^2 - 2x + 1$. Calculate values of y for $x = 0, 1, 2, 3, 4, 5, 6$ from the difference table. Solve by forward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1	0	8	6	0	0
1	1	8	14	6	0	0
2	9	22	20	6	0	
3	31	42	26	6		
4	73	68	32			
5	141	100				
6	241					

$$\Delta^3 y_3 = 6$$

$$\Delta^2 y_3 = 26$$

$$\Delta^2 y_0 = 8$$

Ques 3- Construct a backward difference formula for $y = \log x$ given that $x = 10, 20, 30, 40, 50$, $\nabla^3 \log_{40}$, $\nabla^4 \log_{50} = ?$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1	-0.3010	-0.1249	-0.0737	-0.0505
20	1.3010	-0.1761	-0.0512	-0.0232	
30	1.4771	-0.1249	-0.028		
40	1.6020	-0.0969			
50	1.6989				

$$\nabla^3 \log_{40} = -0.0232$$

$$\nabla^4 \log_{50} = -0.0505$$

* Newton forward interpolation formula -

$$f(a+hu) = f(a) + u \Delta f(a) + \frac{u(u-1)}{2} \Delta^2 f(a) + \frac{u(u-1)(u-2)}{6} \Delta^3 f(a) + \dots + \frac{u(u-1)(u-2)\dots(u-n+1)}{n!} \Delta^n f(a)$$

Ques 1- Find the value of $\sin 52^\circ$ from the given table -

θ	$\sin \theta f(a)$	$\Delta f(a)$	$\Delta^2 f(a)$	$\Delta^3 f(a)$
45°	.7071	→ 589	→ -57	→ -7
50°	.7660	532	-64	
55°	.8192	468		
60°	.8660			

$$h = (50 - 45) = 5$$

$$x = 52^\circ$$

$$a = 45^\circ$$

$$u = \frac{x-a}{h} = 1.4$$

$$\frac{1.4(1.4-1)(1.4-2) \times (-7)}{3}$$

$$f(52) = 7071 + 824.6 + \frac{0.56(-57)}{2} - \frac{0.336(-7)}{6}$$

$$= 7071 + 824.6 - 15.96 + 0.392$$

$$= 7880.032$$

$$\sin 52^\circ \approx 0.7880 \quad \underline{\text{Ans}}$$

Ques- The population of a town in the decimal places was given below estimated the population of the year 1895.

Year (x)	Population (y)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1891	46	20	-5	2	-3
1901	66	15	-3	-1	
1911	81	12	-4		
1921	93	8			
1931	101				

$$h = 10$$

$$u = \frac{x-a}{h} = 0.4$$

$$a = 1891$$

$$x = 1895$$

$$f(1891 + 10 \times 0.4) = 46 + (0.4 \times 20) + \frac{0.4(0.4-1)(-5)}{2} + \frac{0.4(0.4-1)(0.4-2)(2)}{3} + \frac{0.4(0.4-1)(0.4-2)(0.4-3)(-3)}{4}$$

$$f(1895) = 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$f(1895) = 54.85 \quad \underline{\text{Ans}}$$

Ques 3-

From the following table, estimate the number of student marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9	-25	37
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	190				

$h = 10$

$u = \frac{x-a}{h} = \frac{45-40}{10} = 0.5$

$x = 45$

$a = 40$

$$f(40 + 10 \times 0.5) = 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{6} \times (-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} \times 37$$

$$= 31 + 21 + (-1.125) + (-1.5625) + (-1.4455)$$

$$= 47.8672 \text{ Ans.}$$

Ques 4-

Year	1891	1901	1911	1921	1931
Population	46	66	81	93	101

Solve by Newton backward method for the population of the year 1925.

Year	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46	20	-5	2	-3
1901	66	15	-3	-1	
1911	81	12	-4		
1921	93	8			
1931	101				

$$x = 1925$$

$$a = 1931$$

$$h = 10$$

$$u = \frac{x-a}{h} = -0.6$$

$$f(a+hu) = f(a) + u \Delta f(a) + \frac{u(u+1)}{2} \Delta^2 f(a) + \frac{u(u+1)(u+2)}{6} \Delta^3 f(a)$$

$$+ \dots + \frac{u(u+1)(u+2)\dots(u+n+1)}{n!} \Delta^n f(a)$$

$$f[1931+10(0.6)] = 101 - (-0.6) \times 8 + \frac{(-0.6)(-0.6+1)}{2} (-4) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)}{6} (-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24} (3)$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1000$$

$$= 96.836 \text{ Ans.}$$

Ques
★

Newton dividend difference formula :-

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})\Delta^n y_0$$

Ques-

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22	$\frac{30-22}{2-1} = 8$	$\frac{26-8}{4-1} = 6$	$\frac{-3.6-6}{7-1} = -1.6$	
2	30	$\frac{82-30}{4-2} = 26$	$\frac{8-26}{7-2} = -3.6$		
4	82	$\frac{106-82}{7-4} = 8$	$\frac{22-8}{12-4} = 1.75$	$\frac{1.75+3.6}{12-2} = 0.535$	$\frac{0.535+1.6}{12-1} = 0.1940$
7	106	$\frac{216-106}{12-7} = 22$			
12	216				

Ques-

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4	26	15	1
4	56	131	23	
9	711	269		
10	980			

Imp. * Lagrange's Interpolation formula :-

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

Ques1- Using Lagrange interpolation formula find the value of $y(10)$.

	x_0	x_1	x_2	x_3
x	5	6	9	11
f	12	13	14	16
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \cdot 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \cdot 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \cdot 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \cdot 16$$

$$= \left(\frac{-4}{-24} \times 12 \right) + \left(\frac{-5}{15} \times 13 \right) + \left(\frac{-20}{-24} \times 14 \right) + \left(\frac{20}{60} \times 16 \right)$$

$$= 2 - 4.33 + 11.666 + 5.333$$

$$= 14.666 \quad \text{Ans.}$$

Ques2-

	x_0	x_1	x_2	x_3
x	1	2	3	4
$f(x)$	1	8	27	64

find $f(x) = 2.5$

$$y(2.5) = \frac{(2.5-2)(2.5-3)(2.5-4)}{(1-2)(1-3)(1-4)} \cdot 1 + \frac{(2.5-1)(2.5-3)(2.5-4)}{(2-1)(2-3)(2-4)} \cdot 8 +$$

$$\frac{(2.5-1)(2.5-2)(2.5-4)}{(3-1)(3-2)(3-4)} \cdot 27 + \frac{(2.5-1)(2.5-2)(2.5-3)}{(4-1)(4-2)(4-3)} \cdot 64$$

$$= -0.0625 + 4.5 + 15.1875 - 4$$

$$f(0.5) = 15.625 \text{ Ans.}$$

* Gauss Forward Interpolation Formula (Centralized difference formula) :-

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{1 \cdot 2} \Delta^2 y_{-1} + \frac{P(P-1)(P+1)}{1 \cdot 2 \cdot 3} \Delta^3 y_{-1} +$$

$$\frac{P(P-1)(P+1)(P-2)}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^4 y_{-2} + \dots$$

Q1- Interpolate by Gauss forward formula the population of a town for the year 1974 given that -

Year	P	Population (in thousands)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1939	-3	12	3				
1949	-2	15	5	2			
1959	-1	20	7	2	0		
1969	0	27	12	5	3	3	
1974			12	5	-4	-7	
1979	1	39	13	1			
1989	2	52					

Q2- Use forward formula to calculate y_{30} given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$

Solⁿ-

30

x	P	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
21	-2	18.4708				
			-0.6564			
25	-1	17.8144		-0.370		
			-0.7074		-0.0074	
(29)	0	17.1070		-0.0564		-0.0022
			-0.7638		-0.0076	
33	1	16.3432		-0.064		
			-0.8278			
37	2	15.5154				

$$P = \frac{x - x_0}{h} = \frac{30 - 29}{1} = 0.25$$

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{1 \cdot 2} \Delta^2 y_0 + \frac{(P+1)P(P-1)}{1 \cdot 2 \cdot 3} \Delta^3 y_0 + \frac{(P+1)P(P-1)(P-2)}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^4 y_0 + \dots$$

$$y_{30} = 17.1070 + [0.25 \times (-0.7638)] + \frac{(0.25+1)(0.25)(0.25-1)}{1 \cdot 2} (-0.0564) + \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{1 \cdot 2 \cdot 3} (-0.0076) + \dots$$

$$= 17.1070 - 0.19095 + 0.00528 + 0.00029 - 0.000037$$

$$= 16.9215 \quad \Delta 4$$

Q3- Apply a central difference formula to obtain $f(32)$ given that: $f(25) = 0.2707$, $f(30) = 0.3027$
 $f(35) = 0.3386$, $f(40) = 0.3794$

x	P	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
25	-1	0.2707	0.032		
<u>30</u>	0	0.3027	0.0359	0.0039	0.001
35	1	0.3386	0.0408	0.0049	
40	2	0.3794			

$$P = \frac{x - x_0}{h} = \frac{32 - 30}{5} = 0.4$$

$$f(32) = 0.3027 + (0.4 \times 0.0359) + \frac{(0.4)(0.4-1)}{2} \times (0.0039) +$$

$$\frac{(0.4+1)(0.4)(0.4-1)}{3 \cdot 2 \cdot 1} \times (0.001)$$

$$= 0.3027 + 0.01436 + (-0.000488) + (-0.000056)$$

$$= 0.316536 \quad \Delta 1$$

Q4-

x	P	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	-2	1	-2			
2	-1	-1	2	4		
3	0	1	-2	-4	-8	
4	1	-1	2	4	8	16
5	2	1				

$$P = \frac{x - x_0}{h} = \frac{x - 3}{1}$$

$$P = x - 3$$

$$f(P) = 1 + (x-3)(-2) + \frac{(x-3)(x-3-1)(-4)}{2} + \frac{(x-3+1)(x-3)(x-3-1) \times 8}{6} +$$

$$\frac{(x-3+1)(x-3)(x-3-1)(x-3-2)}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{x^2}{6}$$

$$= 1 - 2x + 6 + [(-2x+6)(x-4)] + \frac{4}{3} [(x-2)(x-3)(x-4)] + \frac{2}{3} [(x-2)(x-3)(x-4)(x-5)]$$

$$= 7 - 2x - 2x^2 + 8x + 6x - 24 + \frac{4}{3} [(x^2 - 3x - 2x + 6)(x-4)] + \frac{2}{3} [(x^2 - 5x + 6)(x^2 - 9x + 20)]$$

$$= 7 - 2x - 2x^2 + 14x - 24 + \frac{4}{3} [x^3 - 4x^2 - 5x^2 + 20x + 6x - 24] +$$

$$\frac{2}{3} [x^4 - 9x^3 + 20x^2 - 5x^3 + 45x^2 - 100x + 6x^2 - 54x + 120]$$

$$= -2x^2 + 12x - 17 + \frac{4}{3} [x^3 - 9x^2 + 26x - 24] + \frac{2}{3} [x^4 - 14x^3 + 71x^2 - 154x + 120]$$

$$= -2x^2 + 12x - 17 + \frac{1}{3} (4x^4 - 4x^3 - 36x^2 + 104x - 96 + 2x^4 - 28x^3 - 308x + 142x^2 + 240)$$

$$= -2x^2 + 12x - 17 + \frac{1}{3} (2x^4 - 24x^3 + 106x^2 - 204x + 144)$$

$$= -2x^2 + 12x - 17 + \frac{2}{3}x^4 - 8x^3 + \frac{106}{3}x^2 - \frac{204}{3}x + 48$$

$$= \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31 \quad \checkmark$$

Q5- The values of e^{-x} at $x=1.72$ to 1.76 , Find the value of $e^{-1.7425}$.

x	P	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.72	-2	0.17906	-178			
1.73	-1	0.17728	-176	2		
1.74	0	0.17552	-175	1	-1	
1.75	1	0.17377	-173	2	1	2
1.76	2	0.17204				

$$P = \frac{x - x_0}{h} = \frac{1.7425 - 1.74}{0.01} = 0.25$$

$$y(1.7425) = 17552 + [0.25 \times (-175)] + \left[\frac{0.25(0.25-1) \times 1}{2} \right] +$$

$$\left[\frac{(0.25+1)(0.25)(0.25-1)(1)}{6} \right] + \left[\frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{24} \right]$$

$$= 17552 - 43.75 - 0.09375 - 0.03906 + 0.03417$$

$$= 17508.15136$$

$$e^{-1.7425} = 0.17508 \quad \text{Ans.}$$

* Gauss Backward formula :-

$$f(u) = f(0) + u \Delta f(-1) + \frac{(u+1)u}{2} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{6} \Delta^3 f(-2) +$$

$$+ \frac{(u+2)(u+1)u(u-1)}{24} \Delta^4 f(-2) + \dots$$

Q1- Find the value of $\cos 51^\circ 42'$.

x	$f(x)$ $\cos x$	P	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
50°	0.6428	-2	-135			
51°	0.6293	-1	-136	-1		
(52°)	0.6157	0	-139	-3	-2	4
53°	0.6018	1	-14	-1		
54°	0.5878	2				

$$u = \frac{x - x_0}{h} = \frac{51^\circ 42' - 52^\circ}{1} = \frac{-18'}{60} = -0.3$$

$$f(51^\circ 42') = 6157 + (-0.3)(-136) + \frac{(-0.3+1)(-0.3)(-3)}{2} + \frac{(-0.3+1)(-0.3)(-0.3-1)(-2)}{3} + \frac{(-0.3+2)(-0.3+1)(-3)(-0.3-1)(4)}{4}$$

$$= 6157 + 40.8 + 0.315 - 0.091 + 0.7735$$

$$= 6198.7975$$

$$f(51^\circ 42') = 0.6198 \text{ Ans.}$$

* Stirling formula:-

$$f(u) = f(0) + u \left\{ \frac{\Delta f(0) + \Delta f(-1)}{2} \right\} + \frac{u^2}{2} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3} \left\{ \frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2} \right\} + \frac{\Delta^4 f(-2) + (u+2)(u+1)u(u-1)u-2}{4} \left\{ \frac{\Delta^4 f(-2) + \Delta^4 f(-3)}{2} \right\} + \dots$$

θ	$\tan \theta$	u	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0°	0	-3	875					
15°	0.0875	-2	888	13				
10°	0.1763	-1	916	28	15			
15°	0.2679	0	961	45	17	2		
20°	0.364	1	1023	62	17	0	-2	
25°	0.4663	2	1111	88	26	9	9	
30°	0.5774	3						11

Find the value of $\tan 16^\circ$.

$$u = \frac{16 - 15}{5} = 0.2$$

$$f(16^\circ) = 2679 + (0.2) \left[\frac{961 + 916}{2} \right] + \frac{(0.2)^2}{2} (45) + \frac{(0.2+1)(0.2)(0.2-1)}{6} \left[\frac{17+17}{2} \right] + 0 + \frac{(0.2+2)(0.2+1)(0.2)(0.2-1)(0.2-2)}{120} [-2+9] + \frac{(0.2)^3(0.2^3-1)(0.2^3-4)}{720} (11)$$

$$= 2679 + 187.7 + 0.9 + (-4.624) + 0.022176 + 0.0002323$$

$$= 2862.9983$$

$$\tan 16^\circ = 0.28629983 \quad \text{Ans}$$

Find value of 1.22

x	u	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-2	0	191			
0.5	-1	0.191	150	-41		
1.0	0	0.341	92	-58	-17	27
1.5	1	0.433	44	-48	10	
2.0	2	0.477				

$$u = \frac{1.22 - 1}{0.5} = 0.44$$

$$f(1.22) = 341 + 0.44 \left[\frac{150 + 92}{2} \right] + \frac{(0.44)^2}{2} (-58) + \frac{(0.44+1)(0.44)(0.44-1)}{6} \left[\frac{17+10}{2} \right] + \frac{(0.44)^3(0.44^3-1)(0.44^3-4)}{720} (-27)$$

$$= 341 + 53.44 - 5.6144 + 0.206976 - 0.175024$$

$$= 388.657$$

$$f(1.22) = 0.388657 \quad \text{Ans.}$$

* Bessel's formula :-

$$f(u) = \left\{ \frac{f(0) + f(1)}{2} \right\} + (u - \frac{1}{2}) \Delta f(0) + \frac{u(u-1)}{2} \left\{ \Delta^2 f(0) + \Delta^2 f(1) \right\} \\ + \frac{(u-1)(u-1/2)u}{3} \Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4} \left\{ \frac{\Delta^4 f(-1) + \Delta^4 f(1)}{2} \right\} \\ + \frac{(u-2)(u-1)(u-1/2)u(u+1)}{5} \Delta^5 f(-2) + \dots$$

Quest.

Find the value of 27.4.

u	x	f(x)	$10^3 \Delta f(x)$	$10^3 \Delta^2 f(x)$	$10^3 \Delta^3 f(x)$	$10^3 \Delta^4 f(x)$	$10^3 \Delta^5 f(x)$
-2	25	4.000	-154				
-1	26	3.846	-142	12			
0	27	3.704	-133	9	-3	4	
1	28	3.571	-123	10	-2	-3	-7
2	29	3.448	-115	8			
3	30	3.338					

$$u = \frac{x - x_0}{h} = \frac{27.4 - 27}{1} = 0.4$$

$$10^3 f(0.4) = \left(\frac{3704 + 3571}{2} \right) + (0.4 - 0.5)(-133) + \frac{(0.4)(0.4-1)}{2} (9+10) + \\ \frac{(0.4-1)(0.4-0.5)(0.4)(1)}{6} + \frac{(0.4+1)(0.4)(0.4-1)(0.4-2)}{24} (4-3) \\ + \frac{(0.4-2)(0.4-1)(0.4-0.5)(0.4)(0.4+1)}{120} (-7)$$

$$= 3637.5 + 13.3 - 1.14 + 0.004 + 0.0112 - 0.003136 \\ = 3649.672$$

$$f(27.4) = 3.649672 \quad \text{Ans.}$$

Q2-

Find the value of $f(9)$

u	x	$f(x)$	$10^4 \Delta f(x)$	$10^4 \Delta^2 f(x)$	$10^4 \Delta^3 f(x)$	$10^4 \Delta^4 f(x)$	$10^4 \Delta^5 f(x)$
-2	4	3.5460	15293	1			
-1	6	5.0753	13879	-1414			
0	8	6.4632	12585	-1294	120	5	
1	10	7.7217	11416	-1169	106	-19	
2	12	8.8633	10353	-1063			
3	14	9.8986					

$$u = \frac{9-8}{2} = 0.5$$

$$10^4 f(0.5) = \left(\frac{64632 + 77217}{2} \right) + 0 + \frac{(0.5)(0.5-1)(-1294-1169)}{2} + 0 + \frac{(0.5+1) \cdot 0.5(0.5-1)(0.5-2)(0.5-3)}{24} + 0$$

$$= 70924.5 + 153937.5 - 0.1640625$$

$$= 71078.27344$$

$$f(9) = 7.107827344$$

* Laplace Everett's formula :-

$$f(u) = \left\{ u f(1) + \frac{(u+1)u(u-1)}{3} \Delta^2 f(0) + \frac{(u+2)(u+1)u(u-1)(u-2)}{15} \Delta^4 f(-1) \right\}$$

$$+ \left\{ u f(0) + \frac{(u+1)u(u-1)}{3} \Delta^2 f(-1) + \frac{(u+2)(u+1)u(u-1)(u-2)}{15} \Delta^4 f(-2) \right\}$$

Q1-

Calculate $f(30)$ by Laplace Everett's formula-

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	20	2854	308		
0	28	3162	3926	3618	
1	36	7088	896	-3030	-6648
2	44	7984			

$$u = \frac{30-28}{8} = 0.25$$

$$w = 1 - u = 0.75$$

$$f(0.25) = \left\{ (0.25)(7088) + \frac{(1.25)(.25)(.25-1)(-3030)}{6} \right\} +$$

$$\left\{ (.75)(3162) + \frac{(1.75)(.75)(.75-1)(3618)}{6} \right\}$$

$$= 1772 + 118.359375 + 2371.5 - 197.859375$$

$$= 4064 \text{ Ans.}$$

Q2- Find the value of 27.4.

u	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
-2	25	4.000	-154				
-1	26	3.8446	-142	12			
0	27	3.704	-133	9	-3	4	
1	28	3.571	-123	10	1	-3	-7
2	29	3.448	-115	8	-2		
3	30	3.333					

$$u = 0.4$$

$$w = 1 - 0.4 = 0.6$$

$$10^3 f(0.4) = \left\{ (0.4)(3571) + \frac{(1.4).4(.4-1)(10)}{6} + \frac{(2.4)(1.4).4(.4-1)(.4-2)(-7)}{120} \right\}$$

$$+ \left\{ (0.6)(3704) + \frac{(1.6).6(.6-1)(9)}{6} + \frac{(2.6)(1.6).6(.6-1)(.6-2)(-3)}{120} \right\}$$

$$= 1428.4 - 0.56 - 0.032256 + 2222.4 - 0.576 + 0.0468$$

$$= 3649.678336$$

$$f(27.4) = 3.649678336 \text{ Ans.}$$

UNIT - 3

Numerical differentiation and integration

* Formula for derivative :-

Newton forward formula -

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \dots$$

Diff. this equation in respect to 'u'

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \dots$$

$$u = \frac{x-a}{h}$$

$$\frac{du}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{1}{h} \left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \dots \right]$$

Now i find the formula for the derivative equation

Put the $u=0$

$$\frac{dy}{dx} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots \right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=a} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \dots \right]$$

$$\left(\frac{d^3 y}{dx^3}\right)_{x=a} = \frac{1}{h^3} \left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots \right]$$

Backward formula -

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \dots \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2} \left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \dots \right)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3} \left(\nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \dots \right)$$

Ques 1.

x :	.1	.2	.3	.4
y :	.9975	.9900	.9776	.9604

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
.1	.9975	-75		
.2	.9900	-124	-49	
.3	.9776	-172	-48	1
.4	.9604			

$$\frac{dy}{dx}, x=0.1$$

$$h=0.1$$

$$\frac{dy}{dx} = \frac{1}{h} \left(\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 \right)$$

$$= \frac{1}{0.1} \left(-75 - \frac{1}{2}(-49) + \frac{1}{3}(1) \right)$$

$$= 10 (-75 + 24.5 + 0.3333)$$

$$= -501.666$$

$$\Rightarrow -0.50166$$

Ques 2-

The table given below the velocity v of the body during the time t . Find the acceleration at $t=1.1$.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1	4.6			
1.1	47.7	4.4	-0.2	0.1	
1.2	52.1	4.3	-0.1	0.2	0.1
1.3	56.4	4.4	0.1		
1.4	60.8				

$$h = 0.1$$

$$\frac{dv}{dt} = \frac{1}{h} \left[\Delta v_0 - \frac{1}{2} \Delta^2 v_0 + \frac{1}{3} \Delta^3 v_0 \right]$$

$$= \frac{1}{0.1} \left[4.4 - \frac{1}{2}(-0.1) + \frac{1}{3}(0.2) \right]$$

$$= \frac{1}{0.1} [4.4 + 0.05 + 0.0666]$$

$$= 45.166 \text{ Ans.}$$

★ 1- Trapezoidal Rule ($n=1$)

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

2- Simpson's one-third rule ($n=2$)

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

3- Simpson's three-eighth rule ($n=3$)

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

12 - Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by using

(i) Simpson's one-third rule

(ii) Simpson's three-eighth rule (iii) Trapezoidal rule

x	$f(x)$	
0	1	y_0
1	0.5	y_1
2	0.2	y_2
3	0.1	y_3
4	0.0588	y_4
5	0.03846	y_5
6	0.02702	y_6

$$\begin{aligned} \text{(i)} \quad \int_0^6 \frac{1}{1+x^2} dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{1}{3} [(1 + 0.02702) + 4(0.5 + 0.1 + 0.03846) + 2(0.2 + 0.0588)] \\ &= 1.36617 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \int_0^6 \frac{1}{1+x^2} dx &= \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)] \\ &= \frac{3}{8} [(1 + 0.02707) + 3(0.5 + 0.2 + 0.0588 + 0.03846) + 2(0.1)] \\ &= 1.3570 \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \int_0^6 \frac{1}{1+x^2} dx &= \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)] \\ &= \frac{1}{2} [(1 + 0.02707) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.03846)] \\ &= 1.41077 \end{aligned}$$

✓

Q- Use trapezoidal rule to evaluate $\int_0^1 x^3 dx$ consider 5 sub-intervals

$$\frac{1-0}{5} = 0.2$$

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	0	0.008	0.064	0.216	0.512	1

$$\int_0^1 x^3 dx = \frac{0.2}{2} [(0+1) + 2(0.008 + 0.064 + 0.216 + 0.512)]$$

$$= 0.26 \quad \underline{\text{Ans}}$$

Q- Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ using

(A) Simpson's $\frac{1}{3}$ taking $h = \frac{1}{4}$

(B) Simpson's $\frac{3}{8}$ taking $h = \frac{1}{6}$

(A)

x	0	1/4	2/4	3/4	1
$f(x)$	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

(A)

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{1}{12} [(1 + 0.5) + 4(0.9411 + 0.64) + 2(0.8)]$$

$$= 0.7853$$

(B)

x	0	1	2	3	4	5	1
$f(x)$		6	6	6	6	6	
$f(x)$	1	0.9729	0.9	0.8	0.6923	0.5961	0.5

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{1}{16} [(1+0.5) + 3(0.9729 + 0.9 + 0.6923 + 0.5901) + 2(0.8)]$$

$$= 0.7853 \quad \text{Ans.}$$

Ques-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.2	0.9320	0.0316	-0.0097		
1.3	0.9636	0.0219	-0.0099	-0.0002	
1.4	0.9855	0.0120	-0.0099	0	0.0002
1.5	0.9975	0.0021			
1.6	0.9996				

$y_0 = 0.9320$ and $a = 1.2$, $h = 1.3 - 1.2 = 0.1$
 Newton's forward difference formula,

$$y = y_0 + u \Delta y_0 + \frac{u(u-1)}{2} \Delta^2 y_0 + \dots$$

$$y = 0.9320 + 0.0316u + \frac{u(u-1)}{2} (-0.0097)$$

$$\frac{dy}{du} = 0 + 0.0316 + (2u-1) (-0.0097)$$

At a max, $\frac{dy}{du} = 0$

$$0 = 0.0316 + \frac{(2u-1)}{2} (-0.0097)$$

$$0.0316 = \frac{(2u-1)}{2} (0.0097)$$

$$0.0632 = (2u-1) (0.0097)$$

$$0.0632 = 0.0194u - 0.0097$$

$$u = 3.757$$

$$u = 3.76$$

$$x = a + hu$$

$$x = 1.2 + (0.1)(3.76)$$

$$x = 1.576$$

To find y_{\max} , use backward diff. formula

$$x = x_n + hu$$

$$x = 1.6 + (0.1)u$$

$$1.576 = 1.6 + 0.1u$$

$$u = -0.24$$

$$y(a+hu) = y_n + u \nabla y_n + \frac{u(u+1)}{2} \nabla^2 y_n + \frac{u(u+1)(u+2)}{6} \nabla^3 y_n$$

$$\begin{aligned} y(1.576) &= 0.9996 + (-0.24)(0.0021) + (-0.0535)(-0.0099) \\ &= 0.9996 - 0.0005 + 0.0005 \\ &= 0.9996 \end{aligned}$$

UNIT - 4

Solution of linear equation

* Gauss elimination method -

Ques - By Gauss elimination method solve this equation -

$$5x - y - 2z = 142 \quad \text{--- (1)}$$

$$x - 3y - z = -30 \quad \text{--- (2)}$$

$$2x - y - 3z = -50 \quad \text{--- (3)}$$

Sol:- Step 1 - Dividing (1) by 5

$$x - \frac{y}{5} - \frac{2z}{5} = \frac{142}{5} \quad \text{--- (4)}$$

Now eliminating x from (2) and (3) with the help of (4)

$\begin{array}{r} x - \frac{1}{5}y - \frac{2}{5}z = \frac{142}{5} \\ -x - 3y - z = -30 \\ \hline 3y - \frac{1}{5}y + z - \frac{2}{5}z = \frac{142}{5} + 30 \end{array}$	$\begin{array}{r} x - \frac{1}{5}y - \frac{2}{5}z = \frac{142}{5} \\ x - \frac{1}{2}y - \frac{3}{2}z = -25 \\ \hline \frac{1}{2}y - \frac{1}{5}y + \frac{3}{2}z - \frac{2}{5}z = \frac{142}{5} + 25 \end{array}$
$\frac{14y}{5} + \frac{3z}{5} = \frac{292}{5}$	$\frac{3y}{10} + \frac{11z}{10} = \frac{267}{5}$

Ques - $x + 4y - z = -5$ — (1)

$$x + y - 6z = -12 \quad \text{--- (2)}$$

$$3x - y - z = 4 \quad \text{--- (3)}$$

Sol:- Eliminating x from the eq (2) and eq (3) with the help of eq (1), then we get -

$$\cancel{x} + 4y - z = -5$$

$$\cancel{x} + y - 6z = -12$$

$$\underline{\quad - \quad + \quad +}$$

$$3y + 5z = 7 \quad \text{--- (4)}$$

$$(x + 4y - z = -5) \times 3$$

$$3\cancel{x} + 12y - 3z = -15$$

$$3\cancel{x} - y - z = 4$$

$$\underline{\quad - \quad + \quad + \quad -}$$

$$13y - 2z = -19 \quad \text{--- (5)}$$

Taking eq (4) and (5) eliminating y , then find the value of z .

$$39y + 65z = 91$$

$$\cancel{39y} - 6z = -57$$

$$\underline{\quad - \quad + \quad +}$$

$$71z = 148$$

$$\boxed{z = \frac{148}{71}}$$

Put the value of z in eq (5)

$$3y + 5z = 7$$

$$3y + \frac{740z}{71} = 7$$

$$\boxed{y = \frac{-81}{71}}$$

Now from eq (1)

$$x + 4y - z = -5$$

$$x + \left(\frac{-324}{71}\right) - \frac{148}{71} = -5$$

$$\boxed{x = \frac{117}{71}}$$

$$2x + 3y + 3z = 10 \quad -(2)$$

$$3x - y + 2z = 13 \quad -(3)$$

Sol:- Eliminating x from eq (2) & (3) with the help of (1)

$$2x + 4y + 2z = 6$$

$$3x + 6y + 3z = 9$$

$$2x + 3y + 3z = 10$$

$$3x - y + 2z = 13$$

$$- \quad - \quad - \quad -$$

$$- \quad + \quad - \quad -$$

$$y - z = -4 \quad -(4)$$

$$7y + z = -4 \quad -(5)$$

Taking eq (4) and (5) and eliminating z .

$$y - z = -4$$

$$7y + z = -4$$

$$8y = -8$$

$$\boxed{y = -1}$$

From eq (4)

$$x + 2y + z = -4$$

$$x - z = -4$$

$$\boxed{z = 3}$$

From eq (1)

$$x + 2y + z = 3$$

$$x - 2 + 3 = 3$$

$$\boxed{x = 2}$$

$$x = 2, y = -1, z = 3$$

* Gauss Seidal Integration / Iterative method -

Ques - $20x + y - 2z = 17$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Sol:- We have

$$x = \frac{1}{20} (17 + 2z - y) \quad -(1)$$

$$y = \frac{1}{20} (-18 - 3x + z) \quad -(2)$$

$$z = \frac{1}{20} (25 - 2x + 3y) \quad (3)$$

First approximation - Putting $y=0$ and $z=0$ in eq (1)

$$x_1 = \frac{1}{20} [17 + 0 - 0] = \frac{17}{20} = 0.85$$

Put $x = 0.85$, $z=0$ in eq (2)

$$y_1 = \frac{1}{20} [-18 - 3(0.85) + 0] = -1.0275$$

Put $y = -1.0275$ and $x = 0.85$ in eq (3)

$$z_1 = \frac{1}{20} [25 - 2(0.85) + 3(-1.0275)] = 1.0109$$

Second approximation -

Put $y = -1.0275$ and $z = 1.0109$ in eq (1)

$$x_2 = \frac{1}{20} [17 + 2(1.0109) + 1.0275] = 1.0024$$

Put $x = 1.0024$ and $y, z = 1.0109$ in eq (2)

$$y_2 = \frac{1}{20} [-18 - 3(1.0024) + 1.0109] = -0.9998$$

Put $y = -0.9998$ and $x = 1.0024$ in eq (3)

$$z_3 = \frac{1}{20} [25 - 2(1.0024) + 3(-0.9998)] = 0.9997$$

Third approximation -

Put $y = -0.9998$ and $z = 0.9997$ in eq (1)

$$x_3 = \frac{1}{20} [17 + 0.9998 + 2(0.9997)] = 0.9999$$

Put $x = 0.9999$ and $z = 0.9997$ in eq (2)

$$y_3 = \frac{1}{20} [-18 - 3(0.9999) + 0.9997] = -1$$

Put $x = 0.9999$ and $y = -1$ in eq (3)

$$z_3 = \frac{1}{20} [25 - 2(0.9999) + 3(-1)] = 1.0000$$

Fourth approximation -

Put $y = -1$ and $z = 1$ in eq.(1)

$$x_4 = \frac{1}{20} [17 + 1 + 2(1)] = 1$$

Put $x = 1$ and $z = 1$ in eq.(2)

$$y_4 = \frac{1}{20} [-18 - 3(1) + 1] = -1$$

Put $x = 1$ and $y = -1$ and eq.(3)

$$z_4 = \frac{1}{20} [25 - 2(1) + 3(-1)] = 1$$

Fifth approximation -

$$x_5 = \frac{1}{20} [17 + 1 + 2(1)] = 1$$

$$y_5 = \frac{1}{20} [-18 - 3(1) + 1] = -1$$

$$z_5 = \frac{1}{20} [25 - 2(1) + 3(-1)] = 1$$

$$\boxed{x = 1, y = -1, z = 1} \quad \text{Ans.}$$

* Newton Raphson / Iterative method -

Ques - Find a real root of a equation $x = e^{-x}$.

$$f(x) = x = e^{-x}$$

$$x = \frac{1}{e^x}$$

$$x e^x = 1$$

$$f(x) = x e^x - 1$$

$$f'(x) = x \cdot e^x + e^x \cdot 1 - 0$$

$$f'(x) = e^x (1 + x)$$

$$f(x) = x e^x - 1$$

$$f(0) = -1 \quad (-ve)$$

The root lies between 0 and 1.

$$f(0.5) = -0.1756 \text{ (-ve)}$$

$$f(0.6) = 0.0932 \text{ (+ve)}$$

The root lies between 0.5 and 0.6

By Newton Iterative method -

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(xe^x - 1)}{e^x(1+x)}$$

$$= \frac{xe^x(1+x) - xe^x + 1}{e^x(1+x)}$$

$$= \frac{\cancel{xe^x} + x^2e^x - \cancel{xe^x} + 1}{e^x(1+x)}$$

$$x = \frac{x^2e^x + 1}{e^x(1+x)}$$

$$x_{n+1} = \frac{x_n^2 e^{x_n} + 1}{e^{x_n} (1+x_n)}$$

$n=0$, Taking $x_0 = 0.5$

$$x_1 = \frac{x_0^2 e^{x_0} + 1}{e^{x_0} (1+x_0)}$$

$$= \frac{(0.5)^2 e^{(0.5)} + 1}{e^{(0.5)} (1+0.5)}$$

$$= 0.57102$$

For second iteration, we have $n=1$ and $x_1 = 0.57102$

$$x_2 = \frac{x_1^2 e^{x_1} + 1}{e^{x_1} (1+x_1)}$$

$$= \frac{(0.57102)^2 e^{(0.57102)} + 1}{e^{(0.57102)} (1+0.57102)}$$

$$= 0.56715$$

For third iteration, $n=2$ and $x_2 = 0.56715$

$$x_3 = \frac{x_2^2 e^{x_2} + 1}{e^{x_2} (1+x_2)}$$

$$e^{(0.56715)}(1 + 0.56715)$$

$$= 0.56714$$

So, the correct root is 0.5671 correct to 4 decimal places.

Q-

$$f(x) = x^3 - 3x - 5$$

$$f(0) = -5$$

$$f(1) = -7$$

$$f(2) = -3 \quad (-ve)$$

$$f(3) = 13 \quad (+ve)$$

The root lies between 2 and 3.

$$f(2.5) = 3.125$$

$$f(2.4) = 1.624$$

$$f(2.3) = 0.267 \quad (+ve)$$

$$f(2.2) = -0.952 \quad (-ve)$$

The root lies between 2.2 and 2.3.

$$f'(x) = 3x^2 - 3$$

$$x_{n+1} = x_n - \left[\frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3} \right]$$

$$= \frac{3x_n^3 - 3x_n - x_n^3 + 3x_n + 5}{3x_n^2 - 3}$$

$$x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 3}$$

For first iteration, $n=0$, $x_0 = 2.2$

$$x_1 = \frac{2(2.2)^3 + 5}{3(2.2)^2 - 3} = 2.2826$$

Second iteration, $n=1$, $x_1 = 2.2826$

$$x_2 = \frac{2(2.2826)^3 + 5}{3(2.2826)^2 - 3} = 2.2790$$

Third iteration, $n=2$ and $x_2 = 2.2790$

$$x_3 = \frac{2(2.2790)^3 + 5}{3(2.2790)^2 - 3} = 2.2790$$

The value has matched upto four decimal places. So the correct root is 2.2790.

* Euler Method :- It is the simplest one-step method and has a limited application because of its low accuracy.

In this method, we determine

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

In general, it can be shown that

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Ques:- Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y=1$ for $x=0$. Find y

approximately for $x=0.1$ by Euler method.

Sol:- $x_0=0$, $y_0=1$, $h=0.1$

Then we find the approximate value is -

$n=0$,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{0+1} = y_0 + h f(x_0, y_0)$$

$$y_1 = y_0 + h \left[\frac{y_0 - x_0}{y_0 + x_0} \right]$$

$$y_1 = 1 + 0.1 \left[\frac{1-0}{1+0} \right]$$

$$y_1 = 1.1$$

To find the better accuracy, we obtain by breaking upto the intervals $0-0.1$ into five steps.

We can take the 5 steps $0, 0.02, 0.04, 0.06, 0.08, 0.10$.

x	y
x_0	$y_0 = 1$
x_1	$y_1 = 1.02$
x_2	$y_2 = 1.0392$
x_3	$y_3 = 1.0577$
x_4	$y_4 = 1.0755$
x_5	$y_5 = 1.0927$

$$h = 0.02$$

at $x = 0.02$

$$y_1 = y_0 + h(x_0, y_0)$$

$$y_1 = 1 + 0.02 \left[\frac{1-0}{1+0} \right]$$

$$y_1 = 1.02$$

$x = 0.04$

$$y_2 = y_1 + h(x_1, y_1)$$

$$= 1.02 + 0.02 \left[\frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$y_2 = 1.0392$$

$x = 0.06$

$$y_3 = y_2 + h(x_2, y_2)$$

$$= 1.0392 + 0.02 \left[\frac{1.0392 - 0.04}{1.0392 + 0.04} \right]$$

$$y_3 = 1.0577$$

$x = 0.08$

$$y_4 = y_3 + h(x_3, y_3)$$

$$= 1.0577 + 0.02 \left[\frac{1.0577 - 0.06}{1.0577 + 0.06} \right]$$

$$= 1.0577 + 0.02 \left[\frac{0.9977}{1.1177} \right]$$

$$y_4 = 1.0755$$

$x = 0.1$

$$y_5 = 1.0755 + 0.02 \left[\frac{1.0755 - 0.08}{1.0755 + 0.08} \right]$$

$$y_5 = 1.0927$$

* Runge - Kutta Method :-

1- First order Runge - Kutta method -

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Euler method gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0'$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \dots$$

2- Second order Runge - Kutta method -

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$\Delta y = \frac{1}{2} (k_1 + k_2) \quad \text{Taken in the given order.}$$

Then, $x_1 = x_0 + h$

$$y_1 = y_0 + \Delta y = y_0 + \frac{1}{2} (k_1 + k_2)$$

Similarly,

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + h, y_1 + k_1)$$

$$\Delta y = \frac{1}{2} (k_1 + k_2)$$

Similarly go to next step.

3- Third order Runge - Kutta method -

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$$y_1 = y_0 + \Delta y$$
$$\Delta y = \frac{h}{6} (k_1 + 4k_2 + k_3)$$

Here,

$$k_1 = f(x_0, y_0)$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = f(x_0 + h, y_0 + k_1')$$

$$k_1' = hf(x_0 + h, y_0 + k_1)$$

4- Fourth order Runge - Kutta Method -

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \text{ taken in the given order.}$$

Then,

$$x_1 = x_0 + h$$

$$y_1 = y_0 + \Delta y$$

Similarly,

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$y(0) = 1$ by Runge Kutta method from $x=0$ to 0.4
with $h = .1$

Sol:-

$$f(x, y) = x + y$$

$$x_0 = 0, \quad y_0 = 1$$

$$k_1 = hf(x_0, y_0)$$
$$= h [x_0 + y_0] = .1 [0 + 1] = 0.1$$

$$k_2 = hf \left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right]$$

$$= h [x + y]$$
$$= h \left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2} \right]$$

$$k_2 = .1 \left[0 + \frac{.1}{2} + 1 + \frac{.1}{2} \right] = 0.11$$

$$k_3 = hf \left[x_0 + h, y_0 + k_2 \right]$$

$$= h [x + y]$$
$$= h \left[x_0 + h + y_0 + k_2 \right]$$

$$= .1 \left[0 + .1 + 1 + .11 \right]$$

$$k_3 = 0.1105$$

$$k_4 = hf [x_0 + h, y_0 + k_3]$$

$$= h [x + y]$$

$$= h [x_0 + h + y_0 + k_3]$$

$$= 0.1 [0 + .1 + 1 + 0.1105]$$

$$= 0.12105$$

$$\Delta y = \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6} [.1 + 0.22 + 0.221 + 0.12105]$$

$$\Delta y = 0.11034$$

$$\text{Thus, } x_1 = x_0 + h = 0 + 0.1$$

$$x_1 = 0.1$$

$$y_1 = y_0 + \Delta y$$

$$y_1 = 1 + 0.11034 = 1.11034$$

Now for the second interval we have,

$$k_1 = h f(x_1, y_1)$$

$$= h [x_1 + y_1] = 0.1 [0.1 + 1.11034]$$

$$= 0.121034$$

$$k_2 = h \left[x_1 + \frac{h}{2} + y_1 + \frac{k_1}{2} \right]$$

$$= 0.1 \left[0.1 + \frac{0.1}{2} + 1.11034 + \frac{0.121034}{2} \right]$$

$$= 0.13208$$

$$k_3 = h \left[x_1 + \frac{h}{2} + y_1 + \frac{k_2}{2} \right]$$

$$= 0.1 \left[0.1 + \frac{0.1}{2} + 1.11034 + \frac{0.13208}{2} \right]$$

$$= 0.13263$$

$$k_4 = h [x_1 + h + y_1 + k_3]$$

$$= 0.1 [0.1 + 0.1 + 1.11034 + 0.13263]$$

$$= 0.14429$$

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6} [0.121034 + 0.26416 + 0.26526 + 0.14429]$$

$$\Delta y = 0.132457$$

$$\text{Then, } x_2 = x_1 + h$$

$$= 0.1 + 0.1 = 0.2$$

$$y_2 = y_1 + \Delta y$$

$$= 1.11034 + 0.132457$$

$$= 1.24279$$

$$k_1 = hf(x_2, y_2)$$

$$= h(x_2 + y_2)$$

$$= 0.1(0.2 + 1.24279) = 0.14427$$

$$k_2 = h \left[x_2 + \frac{h}{2} + y_2 + \frac{k_1}{2} \right]$$

$$= 0.1 \left[0.2 + \frac{0.1}{2} + 1.24279 + \frac{0.14427}{2} \right]$$

$$= 0.15710 \quad 0.15649$$

$$k_3 = h \left[x_2 + \frac{h}{2} + y_2 + \frac{k_2}{2} \right]$$

$$= 0.1 \left[0.2 + \frac{0.1}{2} + 1.24279 + \frac{0.15649}{2} \right]$$

$$= 0.15710$$

$$k_4 = h [x_2 + h + y_2 + k_3]$$

$$= 0.1 [0.2 + 0.1 + 1.24279 + 0.15710]$$

$$= 0.16998$$

$$\Delta y = \frac{1}{6} [0.14427 + 2(0.15649) + 2(0.15710) + 0.16998]$$

$$= 0.15690$$

$$x_3 = x_2 + h = 0.2 + 0.1$$

$$= 0.3$$

$$y_3 = y_2 + \Delta y = 1.3997$$

Now, for the fourth interval,

$$k_1 = h [x_3 + y_3] = 0.1 [0.3 + 1.3997]$$

$$= 0.16997$$

$$k_2 = h \left[x_3 + \frac{h}{2} + y_3 + \frac{k_1}{2} \right]$$

$$= 0.1 \left[0.3 + \frac{0.1}{2} + 1.3997 + \frac{0.16997}{2} \right]$$

$$= 0.18346$$

$$k_3 = h \left[x_3 + \frac{h}{2} + y_3 + \frac{k_2}{2} \right]$$

$$= 0.1 \left[0.3 + \frac{0.1}{2} + 1.3997 + \frac{0.18346}{2} \right] = 0.18414$$

$$k_4 = h [x_3 + h + y_3 + k_3]$$

$$= 0.1 [0.3 + 0.1 + 1.3997 + 0.18414] = 0.19838$$

$$\Delta y = \frac{1}{6} [0.16997 + 2(0.18346) + 2(0.18414) + 0.19838]$$

$$= 0.18392$$

$$x_4 = x_3 + h$$

$$= 0.3 + 0.1 = 0.4$$

$$y_4 = y_3 + \Delta y$$

$$= 1.3997 + 0.18392$$

$$y_4 = 1.583625$$

$$\boxed{y_4 = 1.5836}$$

1- Given $\frac{dy}{dx} = y - x$, $y(0) = 2$ find $y(0.1)$ and $y(0.2)$ using

II order method.

$$\frac{dy}{dx} = y - x, \quad x_0 = 0, \quad y_0 = 2, \quad h = 0.1$$

$$k_1 = h f(x_0, y_0)$$

$$= h [y_0 - x_0]$$

$$= 0.1 [2 - 0] = 0.2$$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$

$$= h [y_0 + k_1 - x_0 - h]$$

$$= 0.1 [2 + 0.2 - 0 - 0.1]$$

$$= 0.21$$

$$\Delta y = \frac{1}{2} (k_1 + k_2) = \frac{1}{2} (0.2 + 0.21) = 0.205$$

$$= 0 + 0.1 = 0.1$$

$$y_1 = y_0 + \Delta y \\ = 2 + 0.205 = 2.205$$

Similarly

$$k_1 = h [y_1 - x_1] \\ = 0.1 [2.205 - 0.1] \\ = 0.2105$$

$$k_2 = h [y_1 + k_1 - x_1 - h] \\ = 0.1 [2.205 + 0.2105 - 0.1 - 0.1] \\ = 0.22155$$

$$\Delta y = \frac{1}{2} (0.2105 + 0.22155) = 0.21602$$

$$x_2 = x_1 + h \\ = 0.1 + 0.1 = 0.2$$

$$y_2 = y_1 + \Delta y \\ = 2.205 + 0.21602 = \boxed{2.4210}$$

* Picard Method :-

Ques - $\frac{dy}{dx} = x - y$ with initial condition $y=1$, when $x=0$. Use the picard method to obtain the value y for the point 0.2 .

Sol:- $f(x, y) = x - y$, $x_0 = 0$, $y_0 = 1$

We have first approximation -

$$y^{(1)} = y_0 + \int_0^x f(x, y_0) dx$$

$$y^{(1)} = 1 + \int_0^x (x - y_0) dx$$

$$y^{(1)} = 1 + \int_0^x (x - 1) dx$$

$$y^{(1)} = 1 + \left[\frac{x^2}{2} - x \right]_0^x = 1 + \frac{x^2}{2} - x$$

We have second approximation -

$$\begin{aligned}y^{(2)} &= y_0 + \int_0^x f(x, y_1) dx \\&= 1 + \int_0^x \left\{ x - \left(1 + \frac{x^2 - x}{2} \right) \right\} dx \\&= 1 + \left[\frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{2} \right]_0^x \\y^{(2)} &= 1 + x^2 - x - \frac{x^3}{6}\end{aligned}$$

Third approximation -

$$\begin{aligned}y^{(3)} &= y_0 + \int_0^x f(x, y_2) dx \\&= 1 + \int_0^x (x - y_2) dx \\&= 1 + \int_0^x \left(x - 1 - x^2 + x + \frac{x^3}{6} \right) dx \\&= 1 + \frac{x^2}{2} - x - \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^4}{24} \\y^{(3)} &= 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{24}\end{aligned}$$

Fourth approximation -

$$\begin{aligned}y^{(4)} &= y_0 + \int_0^x (x - y_3) dx \\&= 1 + \int_0^x \left(x - 1 - x^2 + x + \frac{x^3}{3} - \frac{x^4}{24} \right) dx \\&= 1 + \frac{x^2}{2} - x - \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^5}{120} \\y^{(4)} &= 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}\end{aligned}$$

Fifth approximation -

$$y^{(5)} = y_0 + \int_0^x (x - y_4) dx$$

$$y = 1 + \int_0^x \left(2 - x + \frac{x^2}{3} - \frac{x^3}{12} + \frac{x^4}{120} \right) dx$$

$$= 1 + \frac{x^2}{2} - x + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{60} - \frac{x^6}{720}$$

$$y^{(5)} = 1 + \frac{x^2}{2} - x + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{60} - \frac{x^6}{720}$$

When $x=0.2$, we get the value

$$y^{(1)} = 1 + \frac{(0.2)^2}{2} - (0.2) = 0.82$$

$$y^{(2)} = 1 + (0.2)^2 - (0.2) - \frac{(0.2)^3}{6} = 0.8386$$

$$y^{(3)} = 1 + (0.2)^2 - (0.2) - \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 0.8374$$

$$y^{(4)} = 1 + (0.2)^2 - (0.2) - \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} - \frac{(0.2)^5}{120} = 0.8374$$

$$y^{(5)} = 0.8374$$

Thus, $y = 0.837$ when $x = 0.2$.

Ques 2- $\frac{dy}{dx} = 3x + y^2$, $y=1$ at $x=0$. Obtain y for $x=0.1$.

Sol:- $f(x, y) = 3x + y^2$, $x_0=0$, $y_0=1$

First approximation-

$$y^{(1)} = y_0 + \int_0^x f(x, y_0) dx$$

$$= \left\{ y_0 + \int_0^x (3x + y_0^2) dx \right.$$

$$= 1 + \int_0^x (3x + 1) dx$$

$$= 1 + \frac{3x^2}{2} + x$$

Second approximation-

$$y^{(2)} = y_0 + \int_0^x (3x + y_1^2) dx$$

$$= 1 + \int_0^x \left(3x + 1 + x^2 + 2x + \frac{9x^4}{4} + 3x^2 + 3x^3 \right) dx$$

$$= 1 + \frac{3x^2}{2} + x + \frac{x^3}{3} + x^2 + \frac{9x^5}{20} + \frac{3x^3}{3} + \frac{3x^4}{4}$$

$$y^{(2)} = 1 + x + \frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}$$

Third approximation -

$$y^{(3)} = y_0 + \int_0^x \{ 3x + (y_2)^2 \} dx$$

$$y^{(3)} = 1 + \int_0^x \left\{ 3x + \left(1 + x + \frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20} \right)^2 \right\} dx$$

$$y^{(3)} = 1 + \int_0^x \left\{ 3x + 1 + x^2 + 2x + \left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20} \right)^2 + 2(1+x) \left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20} \right) \right\} dx$$

$$y^{(3)} = 1 + \int_0^x \left\{ 1 + 5x + x^2 + \left(\frac{5x^2}{2} + \frac{4x^3}{3} \right)^2 + \left(\frac{3x^4}{4} + \frac{9x^5}{20} \right)^2 + 2 \left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20} \right) + 2(1+x) \left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20} \right) \right\} dx$$

$$y^{(3)} = 1 + \int_0^x \left(1 + 5x + x^2 + \frac{25x^4}{4} + \frac{16x^6}{9} + \frac{20x^5}{3} + \frac{9x^8}{16} + \frac{81x^{10}}{400} + \frac{27x^9}{40} + \frac{15x^6}{4} + \frac{9x^7}{4} + 2x^7 + \frac{18x^8}{15} + 5x^2 + \frac{8x^3}{3} + \frac{3x^4}{2} + \frac{9x^5}{10} + 5x^3 + \frac{8x^4}{3} + \frac{3x^5}{2} + \frac{9x^6}{10} \right) dx$$

$$y^{(3)} = 1 + \int_0^x \left(1 + 5x + 6x^2 + \frac{23x^3}{3} + \frac{125x^4}{12} + \frac{272x^5}{30} + \frac{1157x^6}{180} + \frac{17x^7}{4} + \frac{423x^8}{240} + \frac{27x^9}{40} + \frac{81x^{10}}{400} \right) dx$$

$$y^{(3)} = 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{23x^4}{12} + \frac{25x^5}{12} + \frac{272x^6}{135} + \frac{17x^7}{32} + \frac{47x^8}{240} + \frac{27x^9}{400} + \frac{81x^{10}}{4400}$$

$$y^{(3)} = 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{23x^4}{12} + \frac{25x^5}{12} + \frac{68x^6}{45} + \frac{17x^7}{32} + \frac{47x^8}{240} + \frac{27x^9}{400} + \frac{81x^{10}}{4400}$$

When $x = 0.1$, we get the value

$$y^{(1)} = 1.115$$

$$y^{(2)} = 1.1264$$

$$y^{(3)} = 1.1272$$

Thus, $y = 1.127$ Ans.