BCA 2nd Year

Optimization Techniques

UNIT – 1

Linear Programming

Ques – A tyre factory produces 3 types of tyres T_1, T_2, T_3. Three different types of chemicals say C_1, C_2 and C_3 are required for production. One T_1 needs 2 units of C_1, 3 units of C_3. One T_2 tyre needs 3 units of C_1, 2 units of C_2 and 2 units of C_3 and one T_3 tyre needs 5 units of C_2 and 4 units of C_3. The factory has only one stock of 20 unit of C_1, 25 units of C_2 and 30 units of C_3. Further the profit from the sale of one tyre T_1 is 6 Rs., one tyre of T_2 is 10 Rs., one tyre T_3 is 8 Rs. Assuming that the factory can sell all that it produces formulate a L.P.P. to maximize its profit.

Sol:

	Tyre type T_1	Tyre type T_2	Tyre type T_3	Total units in store
chemical C_1	2	3	0	20
chemical C_2	0	2	5	25
chemical C_3	3	2	4	30
Profit	6 Rs.	10 Rs.	8 Rs.	

Type $T_1 \Rightarrow x_1$

Type $T_2 \Rightarrow x_2$

Type $T_3 \Rightarrow x_3$

Objective function –

$$\max z = 6x_1 + 10x_2 + 8x_3$$

Subject to – $2x_1 + 3x_2 \le 20$

$$2x_2 + 5x_3 \le 25$$

$$3x_1 + 2x_2 + 4x_3 \le 30$$

where $x_1, x_2, x_3 \ge 0$.

Ques- A seller buy some tables and chairs. He has 5000 Rs. to invest and a space to store at most 60 piece. A table cost him 250 Rs. and a chair 50 Rs. He can sell a table at a profit of 50 Rs. and a chair at a profit of 15 Rs. Assuming that he can sell all the pieces that he buy. Prepare a mathematical formulation of this linear programming problem to determine the number of pieces of each type to gain maximum profit.

	Table	Chair	Total units
Cost	250	50	5000
Profit	50	15	

Table $\Rightarrow x_1$

Chair $\Rightarrow x_2$

Objective function -

$$\max Z = 50x_1 + 15x_2$$

Subject to -

$$250x_1 + 50x_2 \leq 5000$$

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★ Simplex method :-

If $\leq \Rightarrow +S$ (Slack variable)

If $\geq \Rightarrow -S$ (Surplus variable)

Ques- $\max z = 4x_1 + 5x_2$

Subject to : $2x_1 + 3x_2 \leq 24$

$2x_1 + x_2 \leq 16$

$x_1, x_2 \geq 0$

Sol: $\max z = 4x_1 + 5x_2 + 0S_1 + 0S_2$

Subject to:

$2x_1 + 3x_2 + S_1 + 0S_2 = 24$

$2x_1 + x_2 + 0S_1 + S_2 = 16$

Key element

	C_j		4	5	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	X_B/X_j
outgoing vector S_1	0	24	2	$\boxed{3}$	1	0	$24/3 = 8 \rightarrow$ min ratio
S_2	0	16	2	1	0	1	$16/1 = 16$
Δ_j	$Z=0$		-4	-5	0	0	

most negative

$\Delta_j = C_B X_j - C_j$

$\Delta_1 = (0,0)(2,2) - 4$
$= -4$

$\Delta_2 = (0,0)(3,1) - 5$
$= -5$

$\Delta_3 = (0,0)(1,0) - 0$
$= 0$

$\Delta_4 = (0,0)(0,1) - 0$
$= 0$

$R_1 \rightarrow R_1/3$, $R_2 \rightarrow R_2 - R_1$

incoming vector Key element

	C_j		4	5	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	
x_2	5	8	2/3	1	1/3	0	$8 \times \frac{3}{2} = 12$
outgoing vector S_2	0	8	$\boxed{4/3}$	0	$-1/3$	1	$8 \times \frac{3}{4} = 6 \rightarrow$ min ratio
Δ_j	$Z=40$		$-2/3$	0	5/3	0	

most negative

$\Delta_1 = (5,0)(2/3, 4/3) - 4$
$= 10/3 - 4 = -2/3$

$\Delta_2 = (5,0)(1,0) - 5$
$= 5-5 = 0$

$\Delta_3 = (5,0)(1/3, -1/3) - 0$
$= 5/3$

$\Delta_4 = (5,0)(0,1) - 0$
$= 0$

$R_1 \rightarrow R_1 - 2/3 R_2$

	C_j		4	5	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_2	5	4	0	1	1/2	-1/2
x_1	4	6	1	0	-1/4	3/4
Δj	z = 44		0	0	3/2	1/2

$\Delta_1 = (5,4)(0,1) - 4$
$= 4-4 = 0$

$\Delta_2 = (5,4)(1,0) - 5$
$= 5-5 = 0$

$\Delta_3 = (5,4)(1/2, -1/4) - 0$
$= 5/2 - 1 = 3/2$

$\Delta_4 = (5,4)(-1/2, 3/4) - 0$
$= -5/2 + 3 = 1/2$

$\Delta j \geq 0$, the solution is optimum.

max $z = 44$

$x_1 = 6$

$x_2 = 4$

Ques – Solve by simplex method –

Max $Z = 4x_1 + 6x_2 + 2x_3$

subject to

$$x_1 + x_2 + x_3 \leq 3$$
$$x_1 + 4x_2 + 7x_3 \leq 9$$
$$x_1, x_2, x_3 \geq 0$$

Sol: $\text{Max } Z = 4x_1 + 6x_2 + 2x_3 + 0S_1 + 0S_2$

subject to: $x_1 + x_2 + x_3 + S_1 + 0S_2 = 3$

$x_1 + 4x_2 + 7x_3 + 0S_1 + S_2 = 9$

$x_1, x_2, x_3 \geq 0$

incoming vector

B.V.	C_j		4	6	2	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	X_B / x_j
S_1	0	3	1	1	1	1	0	$\frac{3}{1} = 3$
outgoing vector S_2	0	9	1	$\boxed{4}$	7	0	1	$\frac{9}{4} = 2.2$ → min ratio
Δ_j	$Z = 0$		-4	-6	-2	0	0	

key element most negative

$\Delta_1 = (0,0)(1,1) - 4 = -4$

$\Delta_2 = (0,0)(1,4) - 6 = -6$

$\Delta_3 = (0,0)(1,7) - 2 = -2$

$\Delta_4 = (0,0)(1,0) - 0 = 0$

$\Delta_5 = (0,0)(0,1) - 0 = 0$

$R_2 \rightarrow R_2/4 \; , \; R_1 \rightarrow R_1 - R_2$

incoming vector

B.V.	C_j		4	6	2	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	X_B / x_j
outgoing vector S_1	0	3/4	$\boxed{3/4}$	0	$-3/4$	1	$-1/4$	$\frac{3}{4} \div \frac{3}{4} = 1$ → min ratio
x_2	6	9/4	1/4	1	7/4	0	1/4	$\frac{9}{4} \div \frac{1}{4} = 9$
Δ_j	$Z = 27/2$		$-5/2$	0	17/2	0	3/2	

most neg

$\Delta_1 = (0,6)(3/4, 1/4) - 4 = 6/4 - 4 = -5/2$

$\Delta_2 = (0,6)(0,1) - 6 = 6 - 6 = 0$

$\Delta_3 = (0,6)(-3/4, 7/4) - 2 = 42/4 - 2 = 17/2$

$\Delta_4 = (0,6)(1,0) - 0 = 0$

$\Delta_5 = (0,6)(-1/4, 1/4) - 0 = \frac{6}{4} = 3/2$

$R_1 \to R_1 \times 4/3$, $R_2 \to R_2 - \frac{1}{4} R_1$

	C_j		4	6	2	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2
x_1	4	1	1	0	-1	3/4	-3/16
x_2	6	2	0	1	2	-1/4	0
Δj	$Z = 16$		0	0	6	3/2	-3/4

$\Delta_1 = (4,6)(1,0) - 4$
$= 4 - 4 = 0$

$\Delta_2 = (4,6)(0,1) - 6$
$= 6 - 6 = 0$

$\Delta_3 = (4,6)(-1,2) - 2$
$= -4 + 12 - 2 = 6$

$\Delta_4 = (4,6)(3/4, -1/4) - 0$
$= 3 - 6/4 = 3/2$

$\Delta_5 = (4,6)(-3/16, 0) - 0$
$= -3/4$

$$\boxed{\begin{array}{l} \text{Max } Z = 16 \\ x_1 = 1,\ x_2 = 2,\ x_3 = 3 \end{array}}$$

Ques- Solve the following L.P.P. by simplex method-

$$\text{Max } Z = 3x_1 + 2x_2$$

subject to :

$$x_1 + x_2 \leq 4$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Sol: Max $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2$

subject to :

$$x_1 + x_2 + S_1 + 0S_2 = 4$$
$$x_1 - x_2 + 0S_1 + S_2 = 2$$
$$x_1, x_2, S_1, S_2 \geq 0$$

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Incoming vector

B.V.	C_j		3	2	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	X_B/x_j
S_1	0	4	1	1	1	0	$\frac{4}{1}=4$
Outgoing vector → S_2	0	2	$\boxed{1}$	-1	0	1	$\frac{2}{1}=2$ → min ratio
Δ_j	Z=0		-3	-2	0	0	

most neg → Key element

$\Delta_1 = (0,0)(1,1)-3$
$= -3$

$\Delta_2 = (0,0)(1,-1)-2$
$= -2$

$\Delta_3 = (0,0)(1,0)-0$
$= 0$

$\Delta_4 = (0,0)(0,1)-0$
$= 0$

$R_1 \to R_1 - R_2$

incoming vector → key element

B.V.	C_j		3	2	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	X_B/x_j
Outgoing vector → S_1	0	2	0	$\boxed{2}$	1	-1	$\frac{2}{2}=1$ → min ratio
x_1	3	2	1	-1	0	1	$\frac{2}{-1}=-2$ (neglect)
Δ_j	Z=6		0	-5	0	3	

most negative

$\Delta_1 = (0,3)(0,1)-3$
$= 3-3 = 0$

$\Delta_2 = (0,3)(2,-1)-2$
$= -3-2 = -5$

$\Delta_3 = (0,3)(1,0)-0$
$= 0$

$\Delta_4 = (0,3)(-1,1)-0$
$= 3$

$R_1 \to R_1/2, \; R_2 \to R_2 + R_1$

B.V.	C_j		3	2	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_2	2	1	0	1	1/2	-1/2
x_1	3	3	1	0	1/2	1/2
Δ_j	Z=11		0	0	5/2	1/2

$\Delta_1 = (2,3)(0,1)-3$
$= 3-3 = 0$

$\Delta_2 = (2,3)(1,0)-2$
$= 2-2 = 0$

$\Delta_3 = (2,3)(1/2, 1/2) - 0$
$= 1 + 3/2 = 5/2$

$\Delta_4 = (2,3)(-1/2, 1/2) - 0$
$= -1 + 3/2 = 1/2$

$$\boxed{\begin{array}{l} Z = 11 \\ x_1 = 3, \quad x_2 = 1 \end{array}}$$

Ques – Max $z = 40x_1 + 60x_2$
subject to: $3x_1 + 3x_2 \le 36$
$5x_1 + 2x_2 \le 60$
$2x_1 + 6x_2 \le 60$

Sol: Max $z = 40x_1 + 60x_2 + 0S_1 + 0S_2 + 0S_3$
subject to:
$3x_1 + 3x_2 + S_1 + 0S_2 + 0S_3 = 36$
$5x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 60$
$2x_1 + 6x_2 + 0S_1 + 0S_2 + S_3 = 60$

incoming vector

		C_j	40	60	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	X_B/x_j
S_1	0	36	3	3	1	0	0	$\frac{36}{3} = 12$
S_2	0	60	5	2	0	1	0	$\frac{60}{2} = 30$
Outgoing vector ← S_3	0	60	2	$\boxed{6}$	0	0	1	$\frac{60}{6} = 10$ → min ratio
Δj	$Z = 0$		-40	-60	0	0	0	

Key element most negative

$\Delta_1 = (0,0,0)(3,5,2) - 40$
$= -40$

$\Delta_2 = (0,0,0)(3,2,6) - 60$
$= -60$

$\Delta_3 = (0,0,0)(1,0,0) - 0$
$= 0$

$\Delta_4 = (0,0,0)(0,1,0) - 0 = 0$

$\Delta_5 = (0,0,0)(0,0,1) - 0 = 0$

$R_1 \to R_1 - 3R_3$, $R_3 \to R_3/6$, $R_2 \to R_2 - 2R_3$

incoming vector → Key element

	C_j		40	60	0	0	0	Ratio X_B/X_j
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	
Outgoing vector → S_1	0	4	$\boxed{2}$	0	1	0	$-1/2$	$\frac{6}{2} = 3$ → min. ratio
S_2	0	40	$13/3$	0	0	1	$-1/3$	$40 \times \frac{3}{13} = 9.2$
x_2	60	10	$1/3$	1	0	0	$1/6$	$\frac{10}{3} = 3.3$
Δj	$Z = 600$		-20	0	0	0	10	

→ most negative

$\Delta_1 = (0,0,60)(2,13/3,1/3) - 40$
$= 20 - 40 = -20$

$\Delta_2 = (0,0,60)(0,0,1) - 60$
$= 60 - 60 = 0$

$\Delta_3 = (0,0,60)(1,0,0) - 0$
$= 0$

$\Delta_4 = (0,0,60)(0,1,0) - 0$
$= 0$

$\Delta_5 = (0,0,60)(-1/2,-1/3,1/6) - 0$
$= 10$

$R_1 \to R_1/2$, $R_2 \to R_2 - \frac{13}{3}R_1$, $R_3 \to R_3 - \frac{1}{3}R_1$

	C_j		40	60	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3
x_1	40	3	1	0	$1/2$	0	$-1/4$
S_2	0	27	0	0	$-13/6$	1	$3/4$
x_2	60	9	0	1	$-1/6$	0	$1/4$
Δj	$Z = 660$		0	0	10	0	5

$\Delta_1 = 40 - 40 = 0$

$\Delta_2 = 60 - 60 = 0$

$\Delta_3 = 20 - 10 = 10$

$\Delta_4 = 0$

$\Delta_5 = -10 + 15 = 5$

$$\boxed{\begin{array}{l} \max Z = 660 \\ x_1 = 3,\ x_2 = 9,\ x_3 = 0 \end{array}}$$

24 Jan 2019

Imp

Ques - Solve the following L.P.P. by simplex method -

$$\max z = 3x_1 + 5x_2 + 4x_3$$

Subject to:

$$2x_1 + 3x_2 \leq 8$$
$$0x_1 + 2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
$$x_1, x_2, x_3 \geq 0$$

Sol:

$$\max z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to:

$$2x_1 + 3x_2 + 0x_3 + S_1 + 0S_2 + 0S_3 = 8$$
$$0x_1 + 2x_2 + 5x_3 + 0S_1 + S_2 + 0S_3 = 10$$
$$3x_1 + 2x_2 + 4x_3 + 0S_1 + 0S_2 + S_3 = 15$$

Incoming vector

Key element

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
Outgoing vector → S_1	0	8	2	$\boxed{3}$	0	1	0	0	$\frac{8}{3} = 2.8$ → min ratio
S_2	0	10	0	2	5	0	1	0	$\frac{10}{2} = 5$
S_3	0	15	3	2	4	0	0	1	$\frac{15}{2} = 7.5$
Δ_j	$z=0$		-3	-5	-4	0	0	0	

most negative

$\Delta_1 = (0,0,0)(2,0,3) - 3$
$= -3$

$\Delta_2 = (0,0,0)(3,2,2) - 5$
$= -5$

$\Delta_3 = (0,0,0)(0,5,4) - 4$
$= -4$

$\Delta_4 = (0,0,0)(1,0,0) - 0$
$= 0$

$\Delta_5 = (0,0,0)(0,1,0) - 0$
$= 0$

$\Delta_6 = (0,0,0)(0,0,1) - 0$
$= 0$

$$R_1 \to \frac{R_1}{3}, \quad R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 2R_1$$

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{8}{3}\times\frac{1}{0}=\infty$
S_2	0	14/3	-4/3	0	$\boxed{5}$	-2/3	-1	0	$14/15 = 0.9 \rightarrow$ min
S_3	0	29/3	5/3	0	4	-2/3	0	1	$29/4 = 2.4$
Δ_j	$Z = 40/3$		1/3	0	-4	5/3	0	0	

most neg.

$\Delta_1 = (5,0,0)(2/3,-4/3,5/3)-3$
$= 10/3 - 3 = 1/3$

$\Delta_2 = (5,0,0)(1,0,0)-5$
$= 5-5 = 0$

$\Delta_3 = (5,0,0)(0,5,4)-4$
$= -4$

$\Delta_4 = (5,0,0)(1/3,-2/3,-2/3)-0$
$= 5/3$

$\Delta_5 = (5,0,0)(0,-1,0)-0$
$= 0$

$\Delta_6 = (5,0,0)(0,0,1)-0$
$= 0$

$R_2 \rightarrow \frac{R_2}{5}$, $R_3 \rightarrow R_3 - 4R_2$

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{8}{3}\times\frac{3}{2} = 4$
x_3	4	14/15	-4/15	0	1	-2/15	1/5	0	$\frac{14}{15}\times\frac{-15}{4} = -\frac{14}{4}$
S_3	0	89/15	$\boxed{41/15}$	0	0	-2/15	-4/5	1	$\frac{89}{15}\times\frac{15}{41} = \frac{89}{41} \rightarrow$ min ratio
Δ_j	$Z = 256/15$		-11/15	0	0	17/15	4/5	0	

most neg.

$\Delta_1 = (5,4,0)(2/3,-4/15,41/15)-3$
$= 10/3 - 16/15 - 3 = -11/15$

$\Delta_2 = (5,4,0)(1,0,0)-5$
$= 5-5 = 0$

$\Delta_3 = (5,4,0)(0,1,0)-4$
$= 4-4 = 0$

$\Delta_4 = (5,4,0)(1/3,-2/15,-2/15)$
$= 5/3 - 8/15 = 17/15$

$\Delta_5 = (5,4,0)(0,1/5,-4/5) = 4/5$

$\Delta_6 = (5,4,0)(0,0,1) = 0$

$R_3 \rightarrow R_3 \times \frac{15}{41}$, $R_2 \rightarrow R_2 + \frac{4}{15} R_3$, $R_1 \rightarrow R_1 - \frac{2}{3} R_3$

	C_j		3	5	4	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_2	5	50/41	0	1	0	15/41	8/41	-10/41
x_3	4	62/41	0	0	1	-6/41	5/41	4/41
x_1	3	89/41	1	0	0	-2/41	-12/41	15/41
Δ_j	$Z = 765/41$		0	0	0	45/41	24/41	11/41

$\Delta_1 = 3-3 = 0$

$\Delta_2 = 5-5 = 0$

$\Delta_3 = 4-4 = 0$

$\Delta_4 = 75/41 - 24/41 - 6/41$
$= 45/41$

$\Delta_5 = 40/41 + 20/41 - 36/41$
$= 24/41$

$\Delta_6 = -50/41 + 16/41 + 45/41$
$= 11/41$

$$\boxed{\begin{array}{c} \max z = \frac{765}{41} \\ x_1 = \frac{89}{41}, \; x_2 = \frac{50}{41}, \; x_3 = \frac{62}{41} \end{array}}$$

Ques – Solve the L.P.P. by simplex method –

Minimize $z = x_1 - 3x_2 + 2x_3$

Subject to :

$$3x_1 - x_2 + 3x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Sol: Converting the minimization into maximization

$$\text{mini } z = x_1 - 3x_2 + 2x_3$$
$$\text{Max } (-z) = -x_1 + 3x_2 - 2x_3$$
$$\text{Max } z^* = -x_1 + 3x_2 - 2x_3$$

Subject to :

$$3x_1 - x_2 + 3x_3 + S_1 + 0S_2 + 0S_3 = 7$$

$-2x_1 + 4x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 12$

$-4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Key element

		C_j		-1	3	-2	0	0	0	Ratio	
	B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_j	
	S_1	0	7	3	-1	3	1	0	0	$\frac{7}{-1} = -7$	
Outgoing vector →	S_2	0	12	-2	$\boxed{4}$	0	0	1	0	$\frac{12}{4} = 3$	→ min ratio
	S_3	0	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.3$	
	Δ_j	$Z^* = 0$		-1	-3	2	0	0	0		

most negative

$\Delta_1 = (0\,0\,0)(3\ -2\ -4) + 1$
$= +1$

$\Delta_2 = (0\,0\,0)(-1\ 4\ 3) - 3$
$= -3$

$\Delta_3 = (0\,0\,0)(3\ 0\ 8) + 2$
$= 2$

$\Delta_4 = (0\,0\,0)(1\ 0\ 0) - 0$
$= 0$

$\Delta_5 = (0\,0\,0)(0\ 1\ 0) - 0$
$= 0$

$\Delta_6 = (0\,0\,0)(0\ 0\ 1) - 0$
$= 0$

$R_2 \to \frac{R_2}{4}$, $R_1 \to R_1 + R_2$, $R_3 \to R_3 - 3R_2$

		C_j		-1	3	-2	0	0	0	Ratio	
	B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_j	
Outgoing vector →	S_1	0	10	$\boxed{5/2}$	0	3	1	1/4	0	$10 \times \frac{2}{5} = 4$	→ min ratio
	x_2	3	3	$-1/2$	1	0	0	1/4	0	$3 \times -\frac{2}{1} = -6$	
	S_3	0	1	$-11/2$	0	8	0	$-3/4$	1	$1 \times -\frac{2}{11} = -2/11$	
	Δ_j	$Z^* = 9$		$-1/2$	0	2	0	3/4	0		

most neg.

$\Delta_1 = (0\ 3\ 0)(5/2\ \ -1/2\ \ -11/2) + 1$
$= -3/2 + 1 = -1/2$

$\Delta_2 = (0\ 3\ 0)(0\ 1\ 0) - 3$
$= 3 - 3 = 0$

$\Delta_3 = (0\ 3\ 0)(3\ 0\ 8) + 2$
$= 2$

$\Delta_4 = (0\ 3\ 0)(1\ 0\ 0) - 0$
$= 0$

$\Delta_5 = (0\ 3\ 0)(1/4\ \ 1/4\ \ -3/4) - 0$
$= 3/4$

$\Delta_6 = (0\ 3\ 0)(0\ 0\ 1) - 0$
$= 0$

$R_1 \to R_1 \times \frac{2}{5}$, $R_2 \to R_2 + \frac{1}{2} R_1$, $R_3 \to R_3 \times R_2$

	C_j		-1	3	-2	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_1	-1	4	1	0	$6/5$	$2/5$	$1/10$	0
x_2	3	5	0	1	$3/5$	$1/5$	$3/10$	0
S_3	0	5	0	0	$24/5$	0	$-9/40$	0
Δ_j	$Z^* = 11$		0	0	$13/5$	$1/5$	$4/5$	0

$\Delta_1 = (-1\ 3\ 0)(1\ 0\ 0) + 1$
$= -1 + 1 = 0$

$\Delta_2 = (-1\ 3\ 0)(0\ 1\ 0) - 3$
$= 3 - 3 = 0$

$\Delta_3 = (-1\ 3\ 0)(6/5\ 3/5\ 24/5) + 2$
$= -6/5 + 9/5 + 2 = 13/5$

$\Delta_4 = (-1\ 3\ 0)(2/5\ 1/5\ 0) - 0$
$= -2/5 + 3/5 = 1/5$

$\Delta_5 = (-1\ 3\ 0)(1/10\ 3/10\ -9/40)$
$= -1/10 + 9/10 = 4/5$

$\Delta_6 = (-1\ 3\ 0)(0\ 0\ 0) - 0$
$= 0$

$$\boxed{\begin{array}{l} \text{Max } z = -11 \\ x_1 = 4,\ x_2 = 5,\ x_3 = 0 \end{array}}$$

★ Big-M-Method (Method of penalty) -

Ques - Mini $z = 4x_1 + 3x_2$

Subject to: $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol: Converting the minimization into maximization

Max $(-z) = z^* = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

Subject to: $2x_1 + x_2 - S_1 + A_1 = 10$

$-3x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 - S_3 + A_2 = 6$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where S_2 is slack variable and S_1, S_3 are surplus variable. A_1 and A_2 are artificial variables.

	C_j		-4	-3	0	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	X_B/x_j
A_1	-M	10	$\boxed{2}$	1	-1	0	0	1	0	$\frac{10}{2} = 5 \to$ min
S_2	0	6	-3	2	0	1	0	0	0	$\frac{6}{-3} = -2$
A_2	-M	6	1	1	0	0	-1	0	1	$\frac{6}{1} = 6$
Δ_j	$Z^* = -16M$		-3M+4	-2M+3	M	0	M	0	0	

most negative

$\Delta_1 = -3M + 4$

$\Delta_2 = -2M + 3$

$\Delta_3 = M$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$\Delta_7 = 0$

$R_1 \to \frac{R_1}{2}, \ R_2 \to R_2 + 3R_1, \ R_3 \to R_3 - R_1$

	C_j		-4	-3	0	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_2	X_B/x_j
x_1	-4	5	1	1/2	-1/2	0	0	0	$5 \times \frac{2}{1} = 10$
S_2	0	21	0	7/2	-3/2	1	0	0	$21 \times \frac{2}{7} = 6$
A_2	-M	1	0	$\boxed{1/2}$	1/2	0	-1	1	$1 \times \frac{2}{1} = 2 \to$ min ratio
Δ_j	$Z^* = -20-M$		0	$-\frac{M}{2}+1$	$-\frac{M}{2}+2$	0	M	0	

most neg.

$\Delta_1 = 0$

$\Delta_2 = -2 - \frac{M}{2} + 3 = \frac{-4 - M + 6}{2} = -\frac{M}{2} + 1$

$\Delta_3 = 2 - \frac{M}{2} = -\frac{M}{2} + 2$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$R_3 \to R_3 \times 2$, $R_2 \to R_2 - \frac{7}{2} R_3$, $R_1 \to R_1 - \frac{1}{2} R_3$

	C_j		-4	-3	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3
x_1	-4	4	1	0	-1	0	1
S_2	0	14	0	0	-5	1	7
x_2	-3	2	0	1	1	0	-2
Δ_j	$Z^* = -22$		0	0	1	0	2

$\Delta_1 = -4 + 4 = 0$

$\Delta_2 = -3 + 3 = 0$

$\Delta_3 = 4 - 3 = 1$

$\Delta_4 = 0$

$\Delta_5 = -4 + 6 = 2$

$\Delta_j \geq 0$, the solution is optimum.

$$\boxed{\begin{array}{l} \max Z^* = -22, \ \text{mini } Z = 22 \\ x_1 = 4, \ x_2 = 2, \ x_3 = 0, \ S_2 = 14 \end{array}}$$

Ques- Solve the L.P.P. by big-M-method -

$$\text{mini } Z = 3x_1 + 8x_2$$

Subject to:

$$x_1 + x_2 = 200$$
$$x_1 \leq 80$$
$$x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Sol: Converting the minimization into maximization

$$\max Z^* = (-Z) = -3x_1 - 8x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

Subject to: $x_1 + x_2 + A_1 = 200$

$$x_1 + S_1 = 80$$
$$x_2 - S_2 + A_2 = 60$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

$S_1 \to$ Slack variable , $S_2 \to$ Surplus variable

$A_1, A_2 \to$ Artificial variables

incoming vector

	C_j		-3	-8	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B/x_j
A_1	-M	200	1	1	0	0	1	0	200
S_1	0	80	1	0	1	0	0	0	∞ (neglect)
Outgoing vector → A_2	-M	60	0	$\boxed{1}$	0	-1	0	1	60 → min ratio
Δ_j	$Z^* = -260M$		-M+3	-2M+8 (most negative)	0	M	0	0	

$\Delta_1 = -M+3$

$\Delta_2 = -2M+8$

$\Delta_3 = 0$

$\Delta_4 = M$

$\Delta_5 = 0$

$\Delta_6 = 0$

$R_1 \rightarrow R_1 - R_3$

	C_j		-3	-8	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_j
A_1	-M	140	1	0	0	1	1	140
S_1	0	80	$\boxed{1}$	0	1	0	0	80 → min ratio
x_2	-8	60	0	1	0	-1	0	∞ (neglect)
Δ_j	$Z^* = -140M - 480$		-M+3	0	0	-M+8	0	

$\Delta_1 = -M+3$

$\Delta_2 = -8+8=0$

$\Delta_3 = 0$

$\Delta_4 = -M+8$

$\Delta_5 = 0$

$R_1 \rightarrow R_1 - R_2$

	C_j		-3	-8	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_j
A_1	-M	60	0	0	-1	$\boxed{1}$	1	60 → min
x_1	-3	80	1	0	1	0	0	∞ (neglect)
x_2	-8	60	0	1	0	-1	0	-60 (neglect)
Δ_j	$Z^* = -60M - 720$		0	0	M-3	-M+8	0	

$\Delta_1 = -3+3=0$

$\Delta_2 = 0$

$\Delta_3 = M-3$

$\Delta_4 = -M+8$

$\Delta_5 = 0$

$R_3 \rightarrow R_3 + R_1$

	C_j		-3	-8	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
S_2	0	60	0	0	-1	1
x_1	-3	80	1	0	1	0
x_2	-8	120	0	1	-1	0
Δ_j	$Z^* = -1200$		0	0	5	0

$\Delta_j \geq 0$ (optimum solution)

$$\boxed{\begin{array}{c} \max Z^* = -1200 \\ \min Z = 1200 \\ x_1 = 80, x_2 = 120, S_2 = 60 \end{array}}$$

Ques - $\text{Max } Z = 4x_1 + 5x_2 - 3x_3$

Subject to: $x_1 + x_2 + x_3 = 10$

$x_1 - x_2 \geq 1$

$2x_1 + 3x_2 + x_3 \leq 40$

$x_1, x_2, x_3 \geq 0$

Sol: $\text{Max } Z = 4x_1 + 5x_2 - 3x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to: $x_1 + x_2 + x_3 + A_1 = 10$

$x_1 - x_2 - S_1 + A_2 = 1$

$2x_1 + 3x_2 + x_3 + S_2 = 40$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

	C_j		4	5	-3	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	X_B/X_j
A_1	-M	10	1	1	1	0	0	1	0	10
S_2	-M	1	$\boxed{1}$	-1	0	-1	0	0	1	1 → min ratio
S_2	0	40	2	3	1	1	1	0	0	20
Δ_j	$Z = -11M$		-2M+4	-5	-M+3	M	0	0	0	

$\Delta_1 = -2M + 4$
$\Delta_2 = -5$
$\Delta_3 = -M + 3$
$\Delta_4 = M$

$\Delta_5 = 0$
$\Delta_6 = 0$
$\Delta_7 = 0$

$R_1 \to R_1 - R_2$, $R_3 \to R_3 - 2R_2$

	C_j		4	5	-3	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	X_B/X_j
A_1	-M	9	0	$\boxed{2}$	1	1	0	1	$\frac{9}{2} = 4.5 \to$ min
x_1	4	1	1	-1	0	-1	0	0	-1 (neglect)
S_2	0	38	0	5	1	2	1	0	$\frac{38}{5} = 7.6$
Δ_j	$z = -9M + 4$		0	$-2M-9$	$-M+3$	$-M-4$	0	0	

$\Delta_1 = 4 - 4 = 0$
$\Delta_2 = -2M - 9$
$\Delta_3 = -M + 3$

$\Delta_4 = -M - 4$
$\Delta_5 = 0$
$\Delta_6 = 0$

$R_1 \to R_1/2$, $R_2 \to R_2 + R_1$, $R_3 \to R_3 - 5R_1$

	C_j		4	5	-3	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2
x_2	5	9/2	0	1	1/2	1/2	0
x_1	4	11/2	1	0	1/2	-1/2	0
S_2	0	31/2	0	0	-3/2	-1/2	1
Δ_j	$z = 89/2$		0	0	15/2	1/2	0

$\Delta_1 = 0$
$\Delta_2 = 0$
$\Delta_3 = \frac{5}{2} + 2 + 3 = \frac{15}{2}$

$\Delta_4 = \frac{5}{2} - 2 = \frac{1}{2}$
$\Delta_5 = 0$

$\Delta_j \geq 0$ (Optimum solution)

$$\boxed{\begin{array}{l} \max z = 89/2 \\ x_1 = \frac{11}{2},\ x_2 = \frac{9}{2},\ S_2 = \frac{31}{2} \end{array}}$$

Ques – Min $z = x_1 + x_2 + 3x_3$

Subject to: $3x_1 + 2x_2 + x_3 \le 3$

$2x_1 + x_2 + 2x_3 \ge 3$

$x_1, x_2, x_3 \ge 0$

Sol: Converting minimization into maximization

Max $z^* = (-z) = -x_1 - x_2 - 3x_3 + 0S_1 + 0S_2 - MA_1$

Subject to: $3x_1 + 2x_2 + x_3 + S_1 = 3$

$2x_1 + x_2 + 2x_3 - S_2 + A_1 = 3$

	C_j		-1	-1	-3	0	0	$-M$	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	X_B/X_j
S_1	0	3	$\boxed{3}$	2	1	1	0	0	$\frac{3}{3} = 1 \rightarrow$ min
A_1	$-M$	3	2	1	2	0	-1	1	$\frac{3}{2} = 1.5$
Δ_j	$z^* = -3M$		$-2M+1$	$-M+1$	$-2M+3$	0	M	0	

$\Delta_1 = -2M+1$

$\Delta_2 = -M+1$

$\Delta_3 = -2M+3$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$R_1 \rightarrow R_1/3$, $R_2 \rightarrow R_2 - 2R_1$

	C_j		-1	-1	-3	0	0	$-M$	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	X_B/X_j
x_1	-1	1	1	$2/3$	$1/3$	$1/3$	0	0	$1 \times \frac{3}{1} = 3$
A_1	$-M$	1	0	$-1/3$	$\boxed{4/3}$	$-2/3$	-1	1	$1 \times \frac{3}{4} = 0.7 \rightarrow$ min
Δ_j	$z^* = -M-1$		0	$\frac{M}{3}+\frac{1}{3}$	$\frac{-4M}{3}+\frac{8}{3}$	$\frac{1}{3}$	M	0	

$\Delta_1 = -1+1 = 0$

$\Delta_2 = -\frac{2}{3} + \frac{M}{3} + 1 = \frac{M}{3} + \frac{1}{3}$

$\Delta_3 = -\frac{1}{3} - \frac{4M}{3} + 3 = \frac{-4M}{3} + \frac{8}{3}$

$\Delta_4 = \frac{-1}{3} + \frac{2}{3} = \frac{1}{3}$

$\Delta_5 = M$

$\Delta_6 = -M + M \neq 0$

$R_2 \rightarrow R_2 \times \frac{3}{4}$, $R_1 \rightarrow R_1 - \frac{1}{3}R_2$

	C_j		-1	-1	-3	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2
x_1	-1	3/4	1	3/4	0	1/2	1/4
x_3	-3	3/4	0	-1/4	1	-1/2	-3/4
Δ_j	$Z^* = -3$		0	1	0	0	2

$\Delta_1 = -1 + 1 = 0$

$\Delta_2 = -\frac{3}{4} + \frac{3}{4} + 1 = 1$

$\Delta_3 = -3 + 3 = 0$

$\Delta_4 = -\frac{1}{2} + \frac{1}{2} = 0$

$\Delta_5 = -\frac{1}{4} + \frac{9}{4} = \frac{8}{4} = 2$

$\max Z^* = -3$

$$\boxed{\begin{array}{l} \min Z = 3 \\ x_1 = \frac{3}{4}, \quad x_3 = \frac{3}{4} \end{array}}$$

★ <u>Two-Phase method</u> –

<u>Ques</u>- Solve the following L.P.P. by two-phase method-

$$\min Z = x_1 + x_2$$

Subject to: $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

<u>Sol:</u> $\max Z^* = (-Z) = -x_1 - x_2$

Subject to: $2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where $S_1, S_2 \rightarrow$ Surplus variable

$A_1, A_2 \rightarrow$ Artificial variable

$x_1 = x_2 = S_1 = S_2 = 0$
$A_1 = 4$
$A_2 = 7$

Phase I – Assign a cost (−1) to Artificial variable and a cost (0) to all other variables.

$$z' = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1 - A_2$$

where z' is a new objective function.

	C_j		0	0	0	0	−1	−1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B/x_j
A_1	−1	4	2	1	−1	0	1	0	4
A_2	−1	7	1	$\boxed{7}$	0	−1	0	1	1 → min ratio
Δ_j	$z' = -11$		−3	−8	1	1	0	0	

most negative

$\Delta_1 = -2-1 = -3$
$\Delta_2 = -1-7 = -8$
$\Delta_3 = 1$
$\Delta_4 = 1$
$\Delta_5 = 0$
$\Delta_6 = 0$

$R_2 \to R_2/7 \quad , \quad R_1 \to R_1 - R_2$

	C_j		0	0	0	0	−1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_j
A_1	−1	3	$\boxed{13/7}$	0	−1	1/7	1	$\frac{21}{13} = 1.$ → min
x_2	0	1	1/7	1	0	−1/7	0	7 ~~→ min~~
Δ_j	$z' = -3$		−13/7	0	1	−1/7	0	

$\Delta_1 = -13/7$
$\Delta_2 = 0$
$\Delta_3 = 1$
$\Delta_4 = -1/7$
$\Delta_5 = 0$

$R_1 \to R_1 \times \frac{7}{13} \quad , \quad R_2 \to R_2 - \frac{1}{7} R_1$

	C_j		0	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	0	21/13	1	0	-7/13	1/13
x_2	0	10/13	0	1	1/13	-2/13
Δ_j	$Z' = 0$		0	0	0	0

$\Delta_1 = 0$ $\quad\quad$ $\Delta_3 = 0$

$\Delta_2 = 0$ $\quad\quad$ $\Delta_4 = 0$

$$\Delta_j \geq 0$$

Phase II – $\quad Z^* = -x_1 - x_2$

	C_j		-1	-1	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	-1	21/13	1	0	-7/13	1/13
x_2	-1	10/13	0	1	1/13	-2/13
Δ_j	$Z^* = -31/13$		0	0	6/13	1/13

$\Delta_1 = -1 + 1 = 0$

$\Delta_2 = 0$

$\Delta_3 = \frac{7}{13} - \frac{1}{13} = \frac{6}{13}$

$\Delta_4 = \frac{-1}{13} + \frac{2}{13} = \frac{1}{13}$

$$\Delta_j \geq 0$$

$$\boxed{\begin{array}{c} Z^* = \frac{-31}{13}, \quad Z = \frac{31}{13} \\ x_1 = \frac{21}{13}, \quad x_2 = \frac{10}{13} \end{array}}$$

Ques- Solve by Two-phase method -

$$\text{Max } z = 2x_1 - x_2 + x_3$$

Subject to: $x_1 + x_2 - 3x_3 \le 8$

$4x_1 - x_2 + x_3 \ge 2$

$2x_1 + 3x_2 - x_3 \ge 4$

$x_1, x_2, x_3 \ge 0$

Sol: $\text{Max } z = 2x_1 - x_2 + x_3$

subject to: $x_1 + x_2 - 3x_3 + S_1 = 8$

$4x_1 - x_2 + x_3 - S_2 + A_1 = 2$

$2x_1 + 3x_2 - x_3 - S_3 + A_2 = 4$

$x_1, x_2, S_1, S_2, A_1, A_2 \ge 0$.

Initial feasable solution

$x_1 = x_2 = S_1 = S_2 = S_3 = 0$

$A_1 = 2$

$A_2 = 4$

Phase I- Assign a cost (-1) to artificial variables and a cost 0 to other variables, then the new objective function -

$$z' = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 + 0S_3 - A_1 - A_2$$

	C_j		0	0	0	0	0	0	-1	-1	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	A_2	X_B/x_j
S_1	0	8	1	1	-3	1	0	0	0	0	8
A_1	-1	2	$\boxed{4}$	-1	1	0	-1	0	1	0	1/2 → min
A_2	-1	4	2	3	-1	0	0	-1	0	1	2
Δ_j	$z' = -6$		-6	-2	0	0	1	1	0	0	

$\Delta_1 = -6$

$\Delta_2 = 1 - 3 = -2$

$\Delta_3 = 0$

$\Delta_4 = 0$

$\Delta_5 = 1$

$\Delta_6 = 1$

$\Delta_7 = 0$

$\Delta_8 = 0$

$R_2 \to \frac{R_2}{4}$, $R_1 \to R_1 - R_2$, $R_3 \to R_3 - 2R_2$

	C_j		0	0	0	0	0	0	-1	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_2	X_B/x_j
S_1	0	15/2	0	5/4	-13/4	1	1/4	0	0	$\frac{15}{2}\times\frac{4}{5} = 6$
x_1	0	1/2	1	-1/4	1/4	0	-1/4	0	0	-ve
A_2	-1	3	0	$\boxed{7/2}$	-3/2	0	1/2	-1	1	$3\times\frac{2}{7} = 6/7 \to$ min
Δ_j	$z' = -3$		0	-7/2	3/2	0	-1/2	1	0	

$\Delta_1 = 0$

$\Delta_2 = -7/2$

$\Delta_3 = 3/2$

$\Delta_4 = 0$

$\Delta_5 = -1/2$

$\Delta_6 = 1$

$\Delta_7 = 0$

$R_3 \to R_3 \times \frac{2}{7}$, $R_2 \to R_2 + \frac{1}{4}R_3$, $R_1 \to R_1 - \frac{5}{4}R_3$

	C_j		0	0	0	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
S_1	0	45/7	0	0	-19/7	1	1/14	5/14
x_1	0	5/7	1	0	1/7	0	-3/14	-1/14
x_2	0	6/7	0	1	-3/7	0	1/7	-2/7
Δ_j	$z' = 0$		0	0	0	0	0	0

Phase II $\quad z = 2x_1 - x_2 + x_3$

	C_j		2	-1	1	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
S_1	0	45/7	0	0	-19/7	1	1/14	5/14	$\frac{45}{7}\times\frac{14}{1} = 90$
x_1	2	5/7	1	0	1/7	0	-3/14	-1/14	-ve
x_2	-1	6/7	0	1	-3/7	0	$\boxed{1/7}$	-2/7	$\frac{6}{7}\times\frac{7}{1} = 6 \to$ min
Δ_j	$z = 4/7$		0	0	-2/7	0	-4/7	1/7	

$\Delta_1 = 7-2-2 = 0$

$\Delta_2 = 0$

$\Delta_3 = \frac{2}{7} + \frac{3}{7} - 1 = -\frac{2}{7}$

$\Delta_4 = 0$

$\Delta_5 = -6/14 - 1/7 = -4/7$

$\Delta_6 = -2/14 + 2/7 = 1/7$

$R_3 \to 7R_3$, $R_1 \to R_1 - R_2 \times \frac{1}{14}$, $R_2 \to R_2 + \frac{3}{14} R_3$

	C_j		2	-1	1	0	0	0	Ratio
BV.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	x_B/x_j
S_1	0	6	0	-1/2	-5/2	1	0	1/2	-ve
x_1	2	2	1	3/2	-1/2	0	0	-1/2	-ve
S_2	0	6	0	7	-3	0	1	-2	-ve
Δ_j	$z = 4$		0	4	-2	0	0	-1	

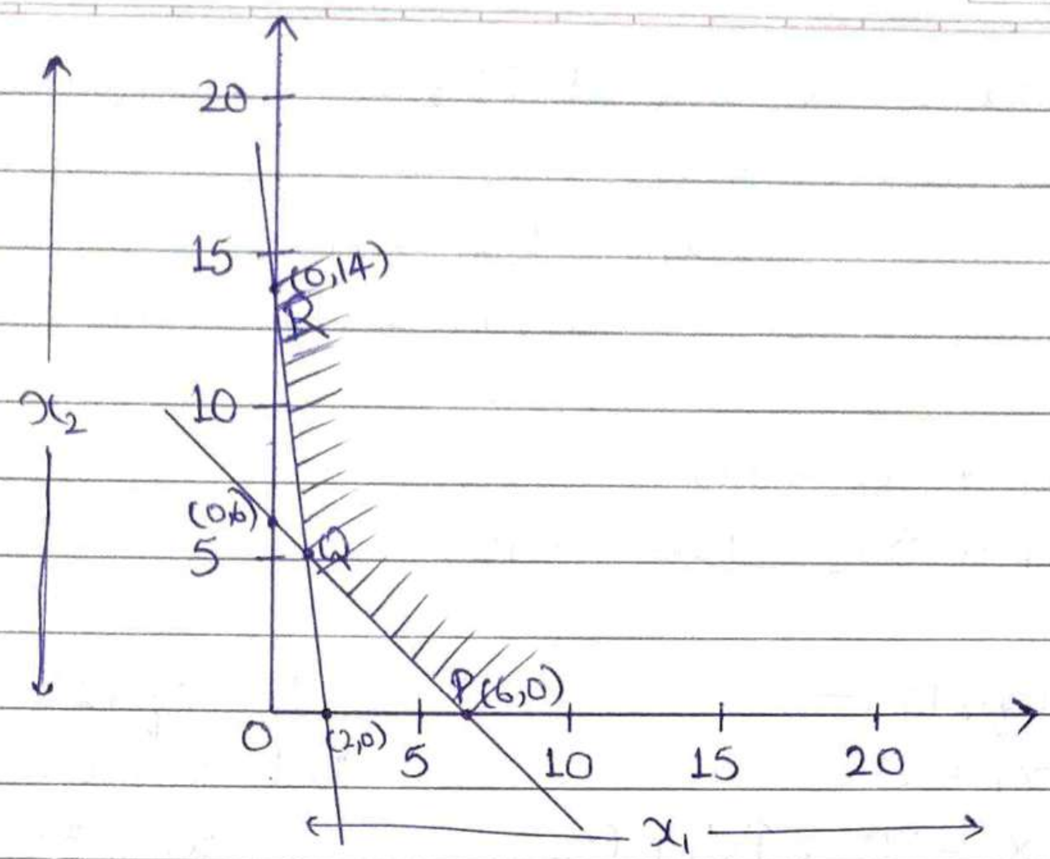
$\Delta_1 = 2-2 = 0$
$\Delta_2 = 3+1 = 4$
$\Delta_3 = -1+1 = -2$
$\Delta_4 = 0$
$\Delta_5 = 0$
$\Delta_6 = -1$

★ Graphical method :-

Ques- Mini $z = 2x_1 + 3x_2$
Subject to: $x_1 + x_2 \geq 6$
$7x_1 + x_2 \geq 14$
$x_1, x_2 \geq 0$

Sol: $x_1 + x_2 = 6$ — (1)
$7x_1 + x_2 = 14$ — (2)
$x_1, x_2 \geq 0$

Put $x_1 = 0$ in eq (1)
$x_2 = 6$, Point $(0, 6)$
Put $x_2 = 0$ in eq (1)
$x_1 = 6$, Point $(6, 0)$
Put $x_1 = 0$ in eq (2)
$x_2 = 14$, Point $(0, 14)$
Put $x_2 = 0$ in eq (2)
$x_1 = 2$, Point $(2, 0)$



$$\begin{array}{rcl} 7x_1 + \cancel{x_2} &=& 14 \\ x_1 + \cancel{x_2} &=& 6 \\ - \quad - & & - \\ \hline 6x_1 &=& 8 \\ x_1 &=& 4/3 \end{array}$$

$x_2 = 6 - x_1$

$x_2 = 6 - \frac{4}{3}$

$x_2 = \frac{14}{3}$

The co-ordinate of Q is $\left(\frac{4}{3}, \frac{14}{3}\right)$

Corner Point	min $z = 2x_1 + 3x_2$	
$P(6,0)$	$2 \times 6 + 0$	12
$Q(4/3, 14/3)$	$2 \times \frac{4}{3} + 3 \times \frac{14}{3}$	$50/3$
$R(0,14)$	$0 + 42$	42

min $z = 12$

$x_1 = 6$

$x_2 = 0$

Ques - $\text{Max } z = 3x_1 + 4x_2$

Subject to: $4x_1 + x_2 \geq 80$

$2x_1 + 5x_2 \leq 180$

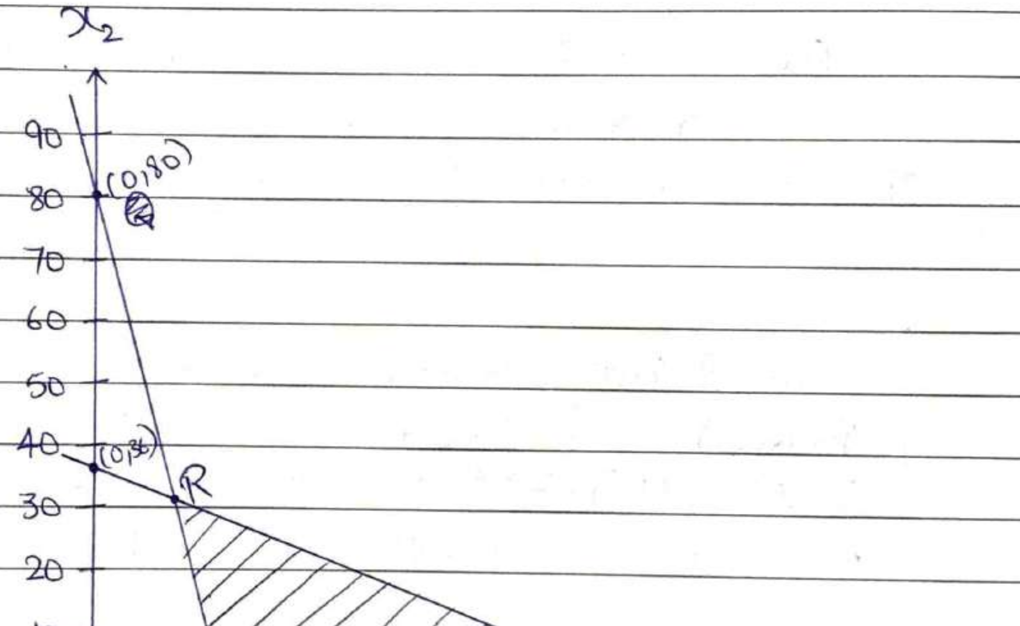
$x_1, x_2 \geq 0$

Sol: $4x_1 + x_2 = 80$ — (1)

$2x_1 + 5x_2 = 180$ — (2)

In eq (1) -
Put $x_1 = 0$
$x_2 = 80$, Point $(0, 80)$
Put $x_2 = 0$
$x_1 = 20$, Point $(20, 0)$

In eq (2) -
Put $x_1 = 0$
$x_2 = 36$, Point $(0, 36)$
Put $x_2 = 0$
$x_1 = 90$, Point $(90, 0)$



$$4x_1 + x_2 = 80$$
$$4x_1 + 10x_2 = 360$$
$$- \quad - \quad -$$

$$-8x_2 = -280$$
$$\boxed{x_2 = 35}$$

Corner Point	$\max z = 3x_1 + 4x_2$
$P(90,0)$	270 → max
$Q(20,0)$	60
$R(5/2, 35)$	$3\times\frac{5}{2} + 4\times 35 = 295/2$

$\max z = 270$

$x_1 = 90, \; x_2 = 0$

Ques-

$x_1 - 2x_2 \le 1$

$x_1 + 2x_2 \ge 3$

$x_1, x_2 \ge 0$

Sol:

$x_1 - 2x_2 = 1 \quad — (1)$

$x_1 + 2x_2 = 3 \quad — (2)$

In eq (1)

$x_1 = 0$

$x_2 = -1/2$ Point $(0, -1/2)$

Put $x_2 = 0$

$x_1 = 1$ Point $(1,0)$

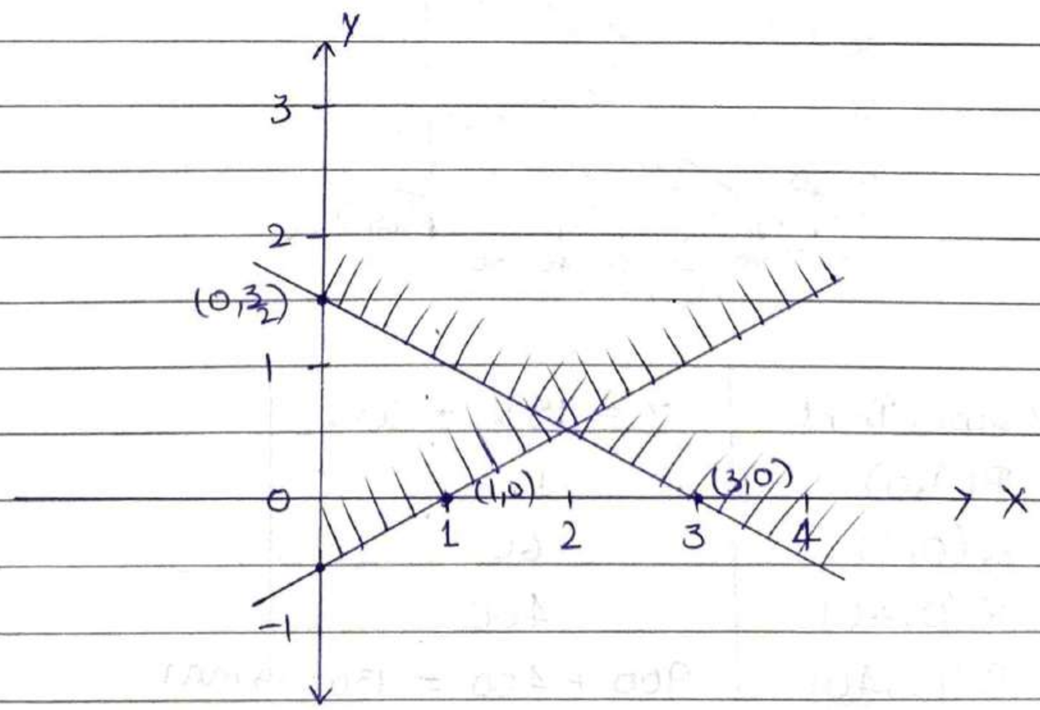
In eq (2)

Put $x_1 = 0$

$x_2 = 3/2$ Point $(0, 3/2)$

Put $x_2 = 0$

$x_1 = 3$ Point $(3,0)$



Ques - Solve by graphical method the following L.P.P. -

$$\text{Max } z = 15x_1 + 10x_2$$

Subject to: $4x_1 + 6x_2 \le 36$

$3x_1 + 0x_2 \le 180$

$0x_1 + 5x_2 \le 200$

$x_1, x_2 \ge 0$

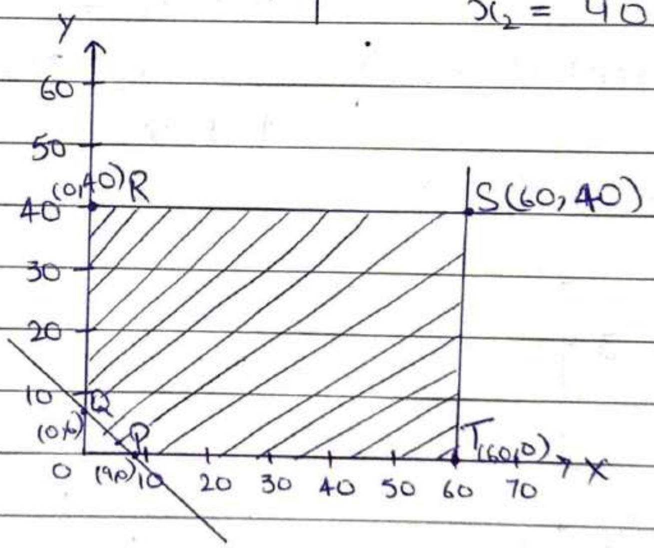
Sol:

$4x_1 + 6x_2 = 36$ — (1)

$3x_1 + 0x_2 = 180$ — (2)

$0x_1 + 5x_2 = 200$ — (3)

In eq (1)	In eq (2)
Put $x_1 = 0$	$3x_1 = 180$
$x_2 = 6$, Point $(0,6)$	$x_1 = 60$, Point $(60,0)$
Put $x_2 = 0$	In eq (3)
$x_1 = 9$, Point $(9,0)$	$5x_2 = 200$
	$x_2 = 40$, Point $(0,40)$



Corner Point	$Z = 15x_1 + 10x_2$
$P(9,0)$	135
$Q(0,6)$	60
$R(0,40)$	400
$S(60,40)$	$900 + 400 = 1300 \rightarrow$ max
$T(60,0)$	900

$\max z = 1300$

$x_1 = 60$, $x_2 = 40$.

Ques- $\max z = 8x_1 + 5x_2$

subject to: $2x_1 + x_2 \geq 8$

$x_1 + x_2 \geq 6$

$x_1 + 3x_2 \geq 9$

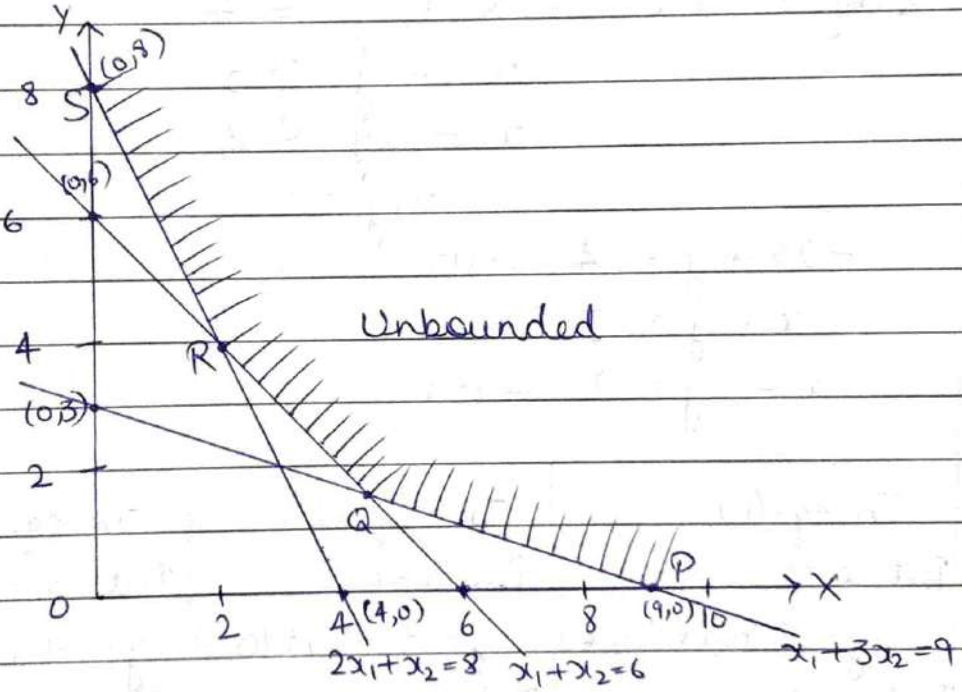
$x_1, x_2 \geq 0$

Sol: $2x_1 + x_2 = 8$ — (1)

$x_1 + x_2 = 6$ — (2)

$x_1 + 3x_2 = 9$ — (3)

In eq (1)	In eq (2)	In eq (3)
Put $x_1 = 0$	Put $x_1 = 0$	Put $x_1 = 0$
$x_2 = 8$, Point $(0,8)$	$x_2 = 6$, Point $(0,6)$	$x_2 = 3$, Point $(0,3)$
Put $x_2 = 0$	Put $x_2 = 0$	Put $x_2 = 0$
$x_1 = 4$, Point $(4,0)$	$x_1 = 6$, Point $(6,0)$	$x_1 = 9$, Point $(9,0)$



$$\cancel{x_1} + x_2 = 6$$
$$\cancel{x_1} + 3x_2 = 9$$
$$- \quad - \quad -$$
$$-2x_2 = -3$$
$$x_2 = \frac{3}{2}$$

$$x_1 + x_2 = 6$$
$$x_1 = 6 - \frac{3}{2}$$
$$x_1 = \frac{9}{2} \Rightarrow Q\left(\frac{3}{2}, \frac{9}{2}\right)$$

$$\begin{aligned} 2x_1 + \cancel{x_2} &= 8 \\ \underset{-}{x_1} \ \underset{-}{+} \ \cancel{x_2} &\underset{-}{=} 6 \\ \hline x_1 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 6 \\ 2 + x_2 &= 6 \\ x_2 &= 4 \Rightarrow R(2,4) \end{aligned}$$

Corner Point	$z = 8x_1 + 5x_2$	
$P(9,0)$	72	$\longrightarrow$ max
$Q(9/2, 3/2)$	$36 + \frac{15}{2} = 87/2$	
$R(2,4)$	$16 + 20 = 36$	
$S(0,8)$	40	

$$\max z = 72$$

$$x_1 = 9, \quad x_2 = 0.$$

Ques-

$$\min z = 3x + 5y$$

subject to:

$$\begin{aligned} -2x + y &\le 4 \\ x + y &\ge 3 \\ x - 2y &\le 2 \\ x, y &\ge 0 \end{aligned}$$

Sol:

$$-2x + y = 4 \quad \text{—(1)}$$

$$x + y = 3 \quad \text{—(2)}$$

$$x - 2y = 2 \quad \text{—(3)}$$

In eq (1)	In eq (2)	In eq (3)
Put $x=0$	Put $x=0$	Put $x=0$
$y=4$, Point $(0,4)$	$y=3$, Point $(0,3)$	$y=-1$, Point $(0,-1)$
Put $y=0$	Put $y=0$	Put $y=0$
$x=-2$, Point $(-2,0)$	$x=3$, Point $(3,0)$	$x=2$, Point $(2,0)$

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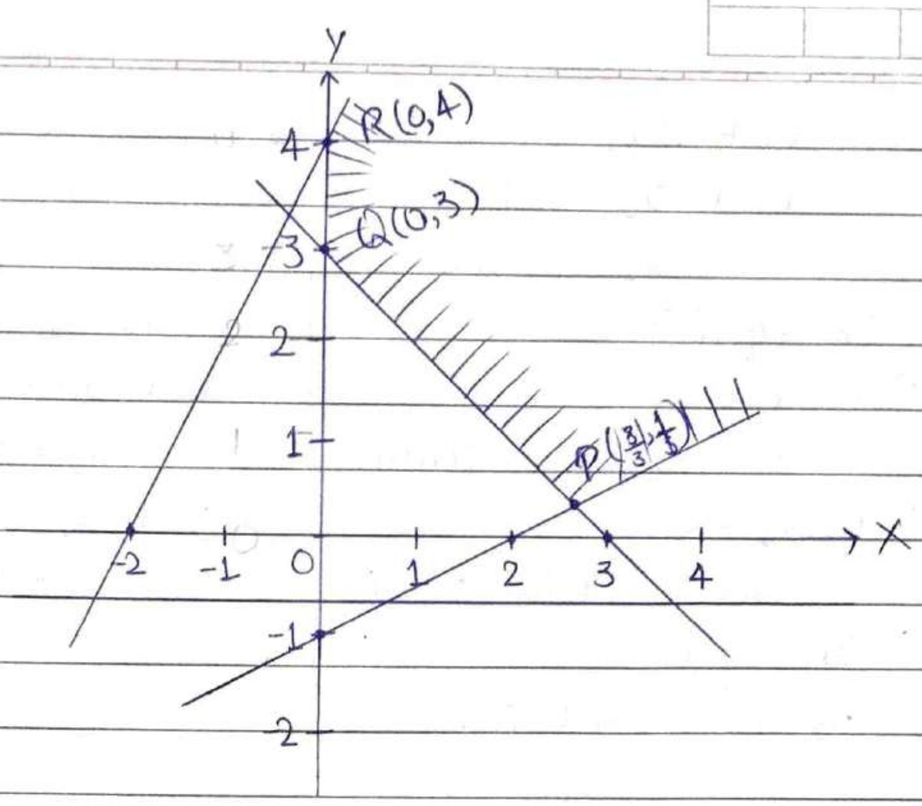


$$x + y = 3$$
$$x - 2y = 2$$
$$- \quad + \quad -$$

$$3y = 1$$
$$y = 1/3$$

$$x + y = 3$$
$$x = 3 - \frac{1}{3}$$
$$x = \frac{8}{3} \Rightarrow P\left(\frac{8}{3}, \frac{1}{3}\right)$$

Corner Point	$Z = 3x + 5y$
P(8/3, 1/3)	29/3 = 9.6 → min
Q (0,3)	15
R (0,4)	20

$$\min Z = 9.6$$
$$x = \frac{8}{3}, \; y = \frac{1}{3}.$$

Ques- Max $Z = 3x_1 + 5x_2$

Subject to: $x_1 + 2x_2 \leq 20$

$x_1 + x_2 \leq 15$

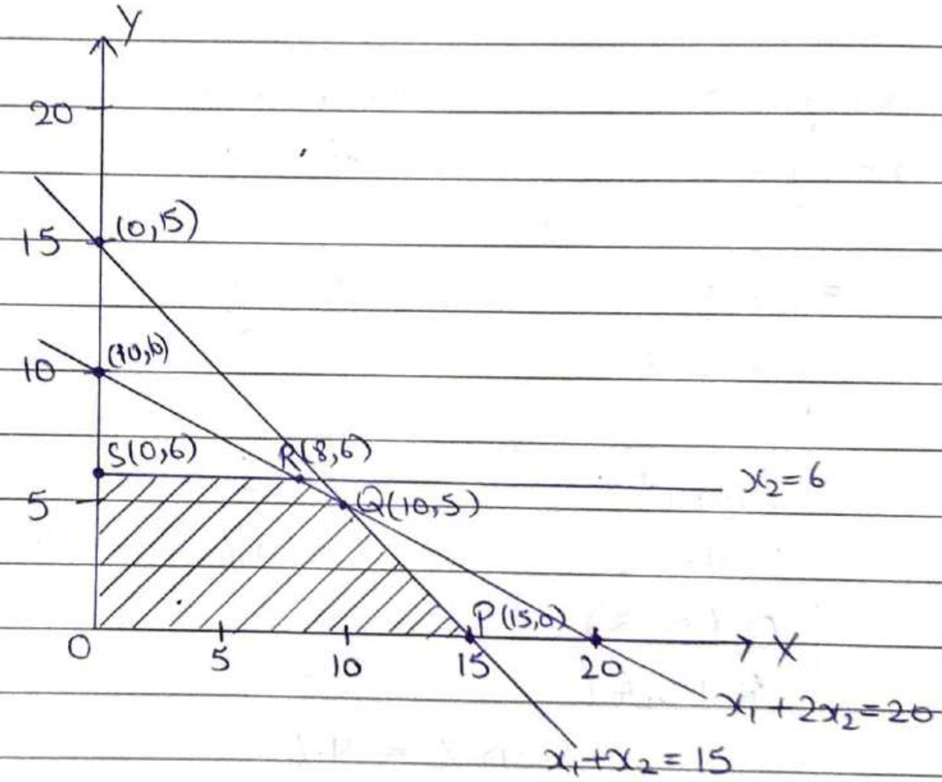
$x_2 \leq 6$

$x_1, x_2 \geq 0$

Sol:
$$x_1 + 2x_2 = 20 \quad -(1)$$
$$x_1 + x_2 = 15 \quad -(2)$$
$$x_2 = 6 \quad -(3)$$

In eq (1)
Put $x_1 = 0$
$x_2 = 10$, Point $(0,10)$
Put $x_2 = 0$
$x_1 = 20$, Point $(20,0)$
In eq (3)
$x_2 = 6$, Point $(0,6)$

In eq (2)
Put $x_1 = 0$
$x_2 = 15$, Point $(0,15)$
Put $x_2 = 0$
$x_1 = 15$, Point $(15,0)$



$$\cancel{x_1} + 2x_2 = 20$$
$$\cancel{x_1} + x_2 = 15$$
$$- \quad - \quad -$$
$$x_2 = 5$$

$$x_1 + x_2 = 15$$
$$x_1 = 15-5$$
$$x_1 = 10$$
$$\Rightarrow Q(10,5)$$

$$x_2 = 6$$
$$x_1 + 2x_2 = 20$$
$$x_1 = 20-12$$
$$x_1 = 8 \qquad \Rightarrow R(8,6)$$

Corner Point	$Z = 3x_1 + 5x_2$
$P(15,0)$	45
$Q(10,5)$	$30 + 25 = 55 \rightarrow$ max
$R(8,6)$	$24 + 30 = 54$
$S(0,10)$	50

$$\max Z = 55$$

$$x_1 = 10, \; x_2 = 5 .$$

* Duality in L.P.P. :-

Ques- $\min Z = 10x_1 + 20x_2$

subject to:

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Sol: Given L.P.P.

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6$$

Primal form -

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$-2x_1 + x_2 \geq -6$$

Matrix form -

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 18 \\ 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

$$Ax \geq b$$

$$\min Z = (10, 20)(x_1, x_2) = cx$$

Dual

$$\text{Max } Z_D = b'y$$
$$\text{max } Z_D = [18, 8, -6]\,[y_1, y_2, y_3]$$
$$\text{max } Z_D = 18y_1 + 8y_2 - 6y_3$$
$$y_1, y_2, y_3 \geq 0$$

$$A'y \leq c'$$
$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$
$$3y_1 + y_2 - 2y_3 \leq 10$$
$$2y_1 + 3y_2 + y_3 \leq 20$$
$$y_1, y_2, y_3 \geq 0 \quad \text{Ans.}$$

Ques- Write the dual of the following L.P.P. —

$$\text{Max } z = 2x_1 + 3x_2 + x_3$$
Subject to: $4x_1 + 3x_2 + x_3 = 6$
$$x_1 + 2x_2 + 5x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

Sol: Change into standard Primal form -

$$4x_1 + 3x_2 + x_3 \leq 6$$
$$-4x_1 - 3x_2 - x_3 \leq -6$$
$$x_1 + 2x_2 + 5x_3 \leq 4$$
$$-x_1 - 2x_2 - 5x_3 \leq -4$$
$$x_1, x_2, x_3 \geq 0$$

Matrix form -

$$\begin{bmatrix} 4 & 3 & 1 \\ -4 & -2 & -1 \\ 1 & 2 & 5 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 6 \\ -6 \\ 4 \\ -4 \end{bmatrix}$$
$$Ax \leq b$$

$$\text{Max } Z = (2\ 3\ 1)\,(x_1\ x_2\ x_3) \Rightarrow cx$$

Duality -

$$\text{Mini } Z_D = b'y$$

$$\text{Mini } Z_D = [6 \ -6 \ 4 \ -4]\,[y_1' \ y_1'' \ y_2' \ y_2'']$$

$$= 6y_1' - 6y_1'' + 4y_2' - 4y_2''$$

$$= 6(y_1' - y_1'') + 4(y_2' - y_2'')$$

$$y_1', y_1'', y_2', y_2'' \geq 0$$

$$A'y \geq c'$$

$$\begin{bmatrix} 4 & -4 & 1 & -1 \\ 3 & -3 & 2 & -2 \\ 1 & -1 & 5 & -5 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2' \\ y_2'' \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$4y_1' - 4y_1'' + y_2' - y_2'' \geq 2 \Rightarrow 4(y_1' - y_1'') + (y_2' - y_2'') \geq 2$$

$$3y_1' - 3y_1'' + 2y_2' - 2y_2'' \geq 3 \Rightarrow 3(y_1' - y_1'') + 2(y_2' - y_2'') \geq 3$$

$$y_1' - y_1'' + 5y_2' - 5y_2'' \geq 1 \Rightarrow (y_1' - y_1'') + 5(y_2' - y_2'') \geq 1$$

Let $y_1 = y_1' - y_1''$

$y_2 = y_2' - y_2''$

$$4y_1 + y_2 \geq 2$$

$$3y_1 + 2y_2 \geq 3$$

$$y_1 + 5y_2 \geq 1$$

$$y_1, y_2 \geq 0$$

$\therefore$ Mini $Z = 6y_1 + 4y_2$ Ans.

Ques- Write the dual of the following L.P.P. -

$$\text{Min } Z = x_1 + x_2 + x_3$$

Subject to: $x_1 - 3x_2 + 4x_3 = 5$

$$x_1 - 2x_2 \leq 3$$

$$2x_2 - x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

Sol: Change into standard primal form -

$$x_1 - 3x_2 + 4x_3 \geq 5$$
$$-x_1 + 3x_2 - 4x_3 \geq -5$$
$$-x_1 + 2x_2 + 0x_3 \geq -3$$
$$0x_1 + 2x_2 - x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Matrix form -

$$\begin{bmatrix} 1 & -3 & 4 \\ -1 & 3 & -4 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 5 \\ -5 \\ -3 \\ 4 \end{bmatrix}$$

$$Ax \geq b$$

$$\text{Mini } Z = (1, 1, 1)(x_1, x_2, x_3) = cx$$

Duality -

$$\text{Max } Z_D = b'y$$
$$\text{Max } Z_D = [5 \ -5 \ -3 \ 4][y_1' \ y_1'' \ y_2 \ y_3]$$
$$= 5y_1' - 5y_1'' - 3y_2 + 4y_3$$
$$= 5(y_1' - y_1'') - 3y_2 + 4y_3$$
$$y_1', y_1'', y_2, y_3 \geq 0$$

$$Ay \leq c'$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 3 & 2 & 2 \\ 4 & -4 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(y_1' - y_1'') - y_2 \leq 1$$
$$-3(y_1' - y_1'') + 2y_2 + 2y_3 \leq 1$$
$$4(y_1' - y_1'') - y_3 \leq 1$$

Let $(y_1' - y_1'') = y_1$

$$y_1 - y_2 \leq 1$$
$$-3y_1 + 2y_2 + 2y_3 \leq 1$$
$$4y_1 - y_3 \leq 1$$

$\therefore$ Max $Z = 5y_1 - 3y_2 + 4y_3$ Ay.

★ What is Operations Research –

"O.R. is the art of giving bad answers to problems which otherwise have worse answers."

– T.L. Saaty

"O.R. is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem."

– C.W. Churchman

★ Essential characteristics of Operations Research –

The significant features of O.R. are given below –

1- Decision making – A major premise of O.R. is that decision-making, irrespective of the situation involved, can be considered as a general systematic process.

2- Scientific approach – O.R. employs scientific methods for the purpose of solving problems. It is a formalised process of reasoning.

3- Objective – O.R. attempts to locate the best or optimal solution to the problem under consideration.

4- Inter-disciplinary Team approach – O.R. is inter-disciplinary in nature and requires a team approach to a solution of the problem. Managerial problems have economic, physical, biological and engineering aspects. This requires a blend of people with expertise in the areas of mathematics, statistics, engineering, economics, management, computer science and so on.

5- Digital computer – Use of digital computer has become an integral part of O.R., approach to decision-making. The computer may be required due to complexity of the model, volume of data required and the computations to be made.

* Advantages and Limitations of a Model :-

Advantages of a model –

1. Through a model, the position under consideration becomes controllable.
2. It help in finding evenues for few research and improvements in a system.
3. It indicates the limitations and scope of an activity.

4. It provide some logical and systematic approach to the problem.

Limitations of a model-

1. Models are only an attempt in understanding operations and should never be considered as absolute in any sense.
2. Validity of any model with regard to corresponding operation can only be verified by carrying the experiment and relevant data characteristics.

* Limitations of Operations Research :-

1- Mathematical models which are essences of O.R. do not take into account quantitative factors which are quite real. All influencing factors which cannot be quantified find no place in mathematical models

2- Mathematical models are applicable to only specific categories of problems.

3- Being the new field there is resistance from the employees to the new proposals.

4- Management, who has to implement the advised proposals, may itself offer a lot of resistance due to conventional thinking.

5- Young enthusiasts, overtaken by its advantages and exactness generally forget that O.R. is meant for men and not that men are meant for it.

* Techniques Used in Operations Research-

The following techniques used in operations research-

1- Linear Programming- It is used in the solution of problems concerned with assignment of personal, blending of materials, distribution and transportation and investment properties.

2- Dynamic programming - It is used in such area as planning, advertising expenditures distributing sales effort and production scheduling etc.

3- Queuing theory - It is used in solving problem concern with traffic, servicing machines subject to break down, air traffic scheduling, production scheduling, hospital operations, determining optimum number of replacement for a group of machines.

4- Inventory theory - In determining when and how much a production or purchase.

5- Game theory - The primary objective of game theory is to develop rational criteria for selecting a strategy.

6- Simulations - The technique of simulation is an important tools of the designer in stimulating airplane flight in a wind tunnel, simulating lines of communication with an organisation chart. With the advent of the high speed digital computer with which to conduct simulated experiments, this technique has become experimental arm of researches.

* Conversion of a L.P.P. into Standard Form :-

Step 1 - Change the linear constraints of inequality type into equations :

This is done with the help of 'slack' or 'surplus' variables. These non-negative variables are added to (or subtracted from) the left hand side of each such constraint. These new variables are added if the constraint is ($\leq$) and are called 'slack variables'. On the other hand, these new variables are subtracted if the constraint is ($\geq$) and are called 'surplus variables'.

For example -

1.) The linear constraint $3x_1 + 4x_2 \leq 20$ is converted to an equation with slack variables S_1 (or x_3) as -

$$3x_1 + 4x_2 + S_1 = 20, \quad \text{where } S_1 \geq 0$$

$$\{\text{or } 3x_1 + 4x_2 + x_3 = 20, \quad \text{where } x_3 \geq 0\}$$

2.) The linear constraint $3x_1 + 7x_2 \geq 28$ is converted to an equation with surplus variable S_2 {or x_4}.

$$3x_1 + 7x_2 - S_2 = 28, \quad \text{where } S_2 \geq 0$$

$$\{\text{or } 3x_1 + 7x_2 - x_4 = 28, \quad \text{where } x_4 \geq 0\}$$

In the objective function the coefficient of slack and surplus variables are shown as zero.

The objective function $Z = 2x_1 + 3x_2$ is written as

$$Z = 2x_1 + 3x_2 + 0S_1 + 0S_2$$

$$\{\text{or } Z = 2x_1 + 3x_2 + 0x_3 + 0x_4\}$$

Hence, the cost of slack or surplus variable is taking zero in objective function.

Step 2 - Make the right ~~had~~ hand element of each constraint non-negative.

This is done by multiplying both sides of the resulting constraint by (-1).

For example-

Consider the constraint $2x_1 - 3x_2 + 4x_3 = -17$

This constraint is changed (in standard form) to

$$-(2x_1 - 3x_2 + 4x_3) = (-1)(-17)$$
$$-2x_1 + 3x_2 - 4x_3 = 17$$

Step 3- Make the unrestricted variables as non-negative: This is done by replacing the unrestricted variable by a difference of two non-negative variables. For example, if x_2 is unrestricted in sign, it can be replaced by -

$$(x_2' - x_2''), \text{ where } x_2' \geq 0,\ x_2'' \geq 0$$
$$\text{by } (x_3 - x_4), \text{ where } x_3 \geq 0,\ x_4 \geq 0$$
$$\text{by } (y_1 - y_2), \text{ where } y_1 \geq 0,\ y_2 \geq 0$$

Step 4- Convert the objective function in maximization form: This is done by changing the sign of the objective function.

For example-

Minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ is equivalent to the expression

Minimize $(-Z) = -c_1x_1 - c_2x_2 - \dots - c_nx_n$

After going through the steps 1 to 4 a given L.P.P. in standard form as -

Optimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0S_1 + \dots + 0S_m$

Subject to
$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \pm S_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \pm S_2 = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \pm S_i = b_i$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \pm S_m = b_m$$

and $x_1, x_2, x_3, \dots, x_n, S_1, \dots S_m \geq 0$

where $b_1, b_2, \dots, b_m \geq 0$

Since standard form involves only equations, we can write it in matrix form as follows -

Optimize $Z = CX$

Such that $AX = b$

and $X \geq 0$

Assignment

on

Linear Programming Problem

Ques 1: Solve the following L.P.P. by simplex method :-

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to : $2x_1 + x_2 \le 18$

$$2x_2 + 5x_3 \le 18$$

$$3x_1 + 2x_2 + 4x_3 \le 25$$

$$x_1, x_2, x_3 \ge 0$$

Sol:

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$$

subject to : $2x_1 + x_2 + 0x_3 + S_1 = 18$

$$2x_2 + 5x_3 + S_2 = 18$$

$$3x_1 + 2x_2 + 4x_3 + S_3 = 25$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

$S_1, S_2, S_3 \rightarrow$ Slack variables

Incoming vector

		C_j	3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
S_1	0	18	2	1	0	1	0	0	18
Outgoing vector ← S_2	0	18	0	$\boxed{2}$	5	0	1	0	9 → min ratio
S_3	0	25	3	2	4	0	0	1	$25/2 = 12.5$
Δ_j	$Z = 0$		-3	-5	-4	0	0	0	

key element

most negative

$\Delta_1 = -3$ $\qquad \Delta_4 = 0$

$\Delta_2 = -5$ $\qquad \Delta_5 = 0$

$\Delta_3 = -4$ $\qquad \Delta_6 = 0$

$$R_2 \rightarrow \frac{R_2}{2}, \quad R_1 \rightarrow R_1 - R_2, \quad R_3 \rightarrow R_3 - 2R_2$$

		C_j	3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
S_1	0	9	2	0	$-5/2$	1	$-1/2$	0	$9/2 = 4.5$
x_2	5	9	0	1	$5/2$	0	$1/2$	0	∞ (neglect)
Outgoing vector ← S_3	0	7	$\boxed{3}$	0	-1	0	-1	1	$7/3 = 2.3$ → min ratio
Δ_j	$Z = 45$		-3	0	$17/2$	0	$5/2$	0	

most negative

$\Delta_1 = 0-3 = -3$ $\quad\quad$ $\Delta_4 = 0$

$\Delta_2 = 5-5 = 0$ $\quad\quad$ $\Delta_5 = 5/2$

$\Delta_3 = 25/2 - 4 = 17/2$ $\quad\quad$ $\Delta_6 = 0$

$R_3 \to \frac{R_3}{3}$, $R_1 \to R_1 - 2R_3$

	C_j		3	5	4	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
S_1	0	13/3	0	0	-11/6	1	1/6	-2/3
x_2	5	9	0	1	5/2	0	1/2	0
x_1	3	7/3	1	0	-1/3	0	-1/3	1/3
Δ_j	$Z=52$		0	0	15/2	0	3/2	1

$\Delta_1 = 3-3 = 0$ $\quad\quad$ $\Delta_4 = 0$

$\Delta_2 = 5-5 = 0$ $\quad\quad$ $\Delta_5 = 5/2 - 1 = 3/2$

$\Delta_3 = \frac{25}{2} - 5 = \frac{15}{2}$ $\quad\quad$ $\Delta_6 = 1$

$\Delta_j \geq 0$ (Optimum solution)

$$\boxed{\begin{array}{l} \text{Max } Z = 52 \\ x_1 = \frac{7}{3},\ x_2 = 9,\ x_3 = 0,\ S_1 = \frac{13}{3} \end{array}}$$

Ques2: Solve the following L.P.P. by Big M method-

$$\text{Min } Z = 4x_1 + 3x_2$$

subject to: $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol: Converting the minimization into maximization-

$\text{Max } Z^* = (-Z)$

$\text{Max } Z^* = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

subject to: $2x_1 + x_2 - S_1 + A_1 = 10$

$-3x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 - S_3 + A_2 = 6$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

where $S_1, S_3 \rightarrow$ Surplus variable

$S_2 \rightarrow$ Slack variable

$A_1, A_2 \rightarrow$ Artificial variables

		C_j	-4	-3	0	0	0	$-M$	$-M$	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	X_B/x_j
Outgoing vector ← A_1	$-M$	10	$\boxed{2}$	1	-1	0	0	1	0	5 → min ratio
S_2	0	6	-3	2	0	1	0	0	0	$-ve$
A_2	$-M$	6	1	1	0	0	-1	0	1	6
Δ_j	$Z^* = -16M$		$-3M+4$ most negative	$-2M+3$	M	0	M	0	0	

$\Delta_1 = -3M + 4$

$\Delta_2 = -2M + 3$

$\Delta_3 = M$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$\Delta_7 = 0$

$R_1 \rightarrow \frac{R_1}{2}, \; R_2 \rightarrow R_2 + 3R_1, \; R_3 \rightarrow R_3 - R_1$

		C_j	-4	-3	0	0	0	$-M$	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_2	X_B/x_j
x_1	-4	5	1	$1/2$	$-1/2$	0	0	0	10
S_2	0	21	0	$7/2$	$-3/2$	1	0	0	6
Outgoing vector ← A_2	$-M$	1	0	$\boxed{1/2}$	$1/2$	0	-1	1	2 → min ratio
Δ_j	$Z^* = -M-20$		0	$1-\frac{M}{2}$ most negative	$2-\frac{M}{2}$	0	M	0	

$\Delta_1 = 0$

$\Delta_2 = \frac{-4}{2} - \frac{M}{2} + 3 = 1 - \frac{M}{2}$

$\Delta_3 = 2 - \frac{M}{2}$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = -M + M = 0$

$R_3 \to R_3 \times 2$, $R_2 \to R_2 - \frac{7}{2}R_3$, $R_1 \to R_1 - \frac{1}{2}R_3$

	C_j		-4	-3	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3
x_1	-4	4	1	0	-1	0	1
S_2	0	14	0	0	-5	1	-7
x_2	-3	2	0	1	1	0	-2
Δj	$Z^* = -22$		0	0	1	0	2

$\Delta j \geq 0$, the solution is optimum.

$$\boxed{\begin{aligned} Z^* &= -22, \quad Z = 22 \\ x_1 &= 4, \quad x_2 = 2, \quad S_2 = 14 \end{aligned}}$$

Ques3: Solve the following L.P.P. by simplex method -

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to:
$$2x_1 + 3x_2 \leq 8$$
$$2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
$$x_1, x_2, x_3 \geq 0$$

Sol:

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$$

subject to:
$$2x_1 + 3x_2 + S_1 = 8$$
$$2x_2 + 5x_3 + S_2 = 10$$
$$3x_1 + 2x_2 + 4x_3 + S_3 = 15$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

where $S_1, S_2, S_3 \to$ Slack variables

		C_j		3	5	4	0	0	0	Ratio	
	B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_j	
Outgoing vector ←	S_1	0	8	2	$\boxed{3}$	0	1	0	0	$\frac{8}{3} = 2.6$	→ min ratio
	S_2	0	10	0	2	5	0	1	0	5	
	S_3	0	15	3	2	4	0	0	1	$\frac{15}{2} = 7.5$	
	Δj	$Z = 0$		-3	-5	-4	0	0	0		

Key element most negative

$\Delta_1 = -3$ $\Delta_4 = 0$

$\Delta_2 = -5$ $\Delta_5 = 0$

$\Delta_3 = -4$ $\Delta_6 = 0$

$R_1 \to \frac{R_1}{3}$, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

			3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
x_2	5	8/3	2/3	1	0	1/3	0	0	∞(neglect)
Outgoing vector ← S_2	0	14/3	-4/3	0	$\boxed{5}$	-2/3	1	0	$\frac{14}{15} = 0.9$ → min ratio
S_3	0	29/3	5/3	0	4	-4/3	0	1	$\frac{29}{12} = 2.4$
Δ_j	Z = 40/3		1/3	0	-4	5/3	0	0	

$\Delta_1 = 10/3 - 1 = 1/3$ most negative $\Delta_4 = 5/3$

$\Delta_2 = 5 - 5 = 0$ $\Delta_5 = 0$

$\Delta_3 = -4$ $\Delta_6 = 0$

$R_2 \to \frac{R_2}{5}$, $R_3 \to R_3 - 4R_2$

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{8}{2} = 4$
x_3	4	14/15	-4/15	0	1	-2/15	1/5	0	$\frac{-14}{4}$ = -ve
S_3	0	89/15	$\boxed{41/15}$	0	0	-2/15	-4/5	1	$\frac{89}{41} = 2.$ → min ratio
Δ_j	Z = 256/15		-11/15	0	0	17/15	4/5	0	

$\Delta_1 = \frac{10}{3} - \frac{16}{15} - 3 = \frac{-11}{15}$ $\Delta_4 = \frac{5}{3} - \frac{8}{15} = \frac{17}{15}$

$\Delta_2 = 5 - 5 = 0$ $\Delta_5 = \frac{4}{5}$

$\Delta_3 = 4 - 4 = 0$ $\Delta_6 = 0$

$R_3 \to R_3 \times \frac{15}{41}$, $R_2 \to R_2 + \frac{4}{15} R_3$, $R_1 \to R_1 - \frac{2}{3} R_3$

	c_j		3	5	4	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_2	5	50/41	0	1	0	15/41	8/41	-10/41
x_3	4	62/41	0	0	1	-6/41	5/41	4/41
x_1	3	89/41	1	0	0	-2/41	-12/41	15/41
Δ_j	Z = 765/41		0	0	0	45/41	24/41	11/41

$\Delta_1 = 3-3 = 0$

$\Delta_2 = 5-5 = 0$

$\Delta_3 = 4-4 = 0$

$\Delta_4 = \frac{75}{41} - \frac{24}{41} - \frac{6}{41} = \frac{45}{41}$

$\Delta_5 = \frac{40}{41} + \frac{20}{41} - \frac{36}{41} = \frac{24}{41}$

$\Delta_6 = -\frac{50}{41} + \frac{16}{41} + \frac{45}{41} = \frac{11}{41}$

$\Delta_j \geq 0$, the solution is optimum.

$$\boxed{\begin{array}{c} \text{Max } Z = \frac{765}{41} \\ x_1 = \frac{89}{41}, \; x_2 = \frac{50}{41}, \; x_3 = \frac{62}{41} \end{array}}$$

Ques 4:- Solve the following L.P.P. by two-phase method -

$$\text{Min } Z = x_1 + x_2$$

subject to: $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Sol: Converting the minimization into maximization

$$\text{Max } Z^* = (-Z) = -x_1 - x_2$$

subject to: $2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where $S_1, S_2 \rightarrow$ Surplus variable

$A_1, A_2 \rightarrow$ Artificial variable

$x_1 = x_2 = S_1 = S_2$
$A_1 = 4$
$A_2 = 7$

Phase I – Assign a cost (−1) to Artificial variables and a cost (0) to all other variables.

$$Z' = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1 - A_2$$

	C_j		0	0	0	0	−1	−1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	C_B/x_j
A_1	−1	4	2	1	−1	0	1	0	4
A_2	−1	7	1	$\boxed{7}$	0	−1	0	1	1 → min ratio
Δ_j	$Z' = -11$		−3	−8	1	1	0	0	

$\Delta_1 = -2-1 = -3$ $\Delta_4 = 1$
$\Delta_2 = -1-7 = -8$ $\Delta_5 = 0$
$\Delta_3 = 1$ $\Delta_6 = 0$

$R_2 \to \frac{R_2}{7}$, $R_1 \to R_1 - R_2$

	C_j		0	0	0	0	−1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	C_B/x_j
A_1	−1	3	$\boxed{13/7}$	0	−1	1/7	1	$\frac{21}{13} = 1.$ → min ratio
x_2	0	1	1/7	1	0	−1/7	0	7
Δ_j	$Z' = -3$		−13/7	0	1	−1/7	0	

$\Delta_1 = -13/7$ $\Delta_4 = -1/7$
$\Delta_2 = 0$ $\Delta_5 = 0$
$\Delta_3 = 1$

$R_1 \to R_1 \times \frac{7}{13}$, $R_2 \to R_2 - \frac{1}{7} R_1$

	C_j		0	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	0	21/13	1	0	-7/13	1/13
x_2	0	10/13	0	1	1/13	-2/13
Δ_j	$Z'=0$		0	0	0	0

$\Delta_1 = 0 \qquad \Delta_3 = 0$

$\Delta_2 = 0 \qquad \Delta_4 = 0$

$$\Delta_j \geq 0$$

<u>Phase II</u> — $Z^* = -x_1 - x_2$

	C_j		-1	-1	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	-1	21/13	1	0	-7/13	1/13
x_2	-1	10/13	0	1	1/13	-2/13
Δ_j	$Z^* = -31/13$		0	0	6/13	1/13

$\Delta_1 = -1 + 1 = 0 \qquad \Delta_4 = -\frac{1}{13} + \frac{2}{13} = \frac{1}{13}$

$\Delta_2 = 0$

$\Delta_3 = \frac{7}{13} - \frac{1}{13} = \frac{6}{13}$

$\Delta_j \geq 0$, the solution is optimum

$$\boxed{Z^* = -\frac{31}{13}, \quad Z = \frac{31}{13}}$$

$$\boxed{x_1 = \frac{21}{13}, \quad x_2 = \frac{10}{13}}$$

Ques 5:- Write the dual of the following problem —

$$\text{Min } Z = 2x_2 + 5x_3$$

subject to: $x_1 + x_2 \geq 2$

$$2x_1 + x_2 + 6x_3 \leq 6$$

$$x_1 - x_2 + 3x_3 = 4$$

$$x_1, x_2, x_3 \geq 0$$

Sol: Changing into Standard Primal form-

$$x_1 + x_2 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1, x_2, x_3 \geq 0$$

Matrix form-

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -6 \\ 4 \\ -4 \end{bmatrix}$$

$$A x \geq b$$

$$\text{Mini } Z = (0 \ 2 \ 5)(x_1 \ x_2 \ x_3) = cx$$

Duality -

$$\text{Max } Z_D = b'y$$
$$\text{Max } Z_D = [2 \ -6 \ 4 \ -4][y_1, y_2, y_3', y_3'']$$
$$= 2y_1 - 6y_2 + 4(y_3' - y_3'')$$
$$y_1, y_2, y_3', y_3'' \geq 0$$

$$A'y \leq c'$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 0 & -6 & 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3' \\ y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$y_1 - 2y_2 + (y_3' - y_3'') \leq 0$$
$$y_1 - y_2 - (y_3' - y_3'') \leq 2$$
$$-6y_2 + 3(y_3' - y_3'') \leq 5$$

Let $(y_3' - y_3'') = y_3$

$$y_1 - 2y_2 + y_3 \leq 0$$
$$y_1 - y_2 - y_3 \leq 2$$
$$-6y_2 + 3y_3 \leq 5$$

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$$\text{Max } Z = 2y_1 - 6y_2 + 4y_3$$

Ans.

* Transportation Problem :-
Transportation problem is a special kind of linear programming problem in which goods are transported from a set of sources to a set of destination subject to supply and demand of the source and destination. respectively such that the total cost of transportation problem is minimize.

Let n = The number of sources
m = The number of destination
a_i = The supply at the source i.
d_j = The demand at the destination j.
C_{ij} = The cost of transportation per unit from i^{th} source to j^{th} destination.
x_{ij} = The number of units to be transported from the source i^{th} to the j^{th} destination.

* Mathematical formulation of the Transportation Problem-
Mathematically, the problem may be stated as follows-

$$\text{Min}(z) = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}\, x_{ij}$$

Subject to : $\sum_{i=1}^{m} x_{ij} = a_i$ for $i = 1,2,3 \ldots m$

$$\sum_{j=1}^{n} x_{ij} = b_j \text{ for } j = 1,2,3 \ldots n$$

For a feasable solution to exist, it is necessary that total supply equals total requirement.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Destination

	W_1	W_2 ---	W_j ----	W_n	Supply
	C_{11}	C_{12} ···	C_{1j} ···	C_{1n}	a_1
	C_{21}	C_{22} ···	C_{2j} ···	C_{2n}	a_2
	⋮	⋮	⋮	⋮	⋮
Origin	C_{i1}	C_{i2} ···	C_{ij} ···	C_{in}	a_i
	⋮	⋮	⋮	⋮	⋮
	C_{m1}	C_{m2} ···	C_{mj} ···	C_{mn}	a_m
Demand	b_1	b_2 ---	b_j ---	b_m	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

★ North-West corner cell method :-

Step 1- Find the minimum of the supply and demand value with respect to the current north-west corner cell of the cost matrix.

Step 2- Allocate the minimum value to the current north-west corner cell and subtract with minimum from the supply and demand value of the current north-west corner cell.

Step 3- Check whether exactly one of the row or column correspondence to north-west corner cell has 0 supply or demand respectively. If so, go to step 4 or step 5.

Step 4- Delete the row or column of the current north-west cell which has 0 supply or deman and go to step 6.

Step 5- Delete both the row and column with respect to the current north-west corner cell.

Step 6- Check whether exactly one row or column left out, if yes go to step 7 else go to step 1.

Step 7 - Match the supply or demand of the row or column with the remaining demands or supply of the undeleted columns or row.

Step 8 - Go to phase 2 for optimization of solution obtained above.

Ques —

Destination

Source	1	2	3	4	Supply
1	3	1	7	4	300
2	2	6	5	9	400
3	8	3	3	2	500
Demand	250	350	400	200	1200

Sol:

Destination

Source	1	2	3	4	Supply
1	[250] 3	1	7	4	300/50
2	2	6	5	9	400
3	8	3	3	2	500
Demand	250/0	350	400	200	1200

Source	2	3	4	Supply
1	[50] 1	7	4	50/0
2	6	5	9	400
3	3	3	2	500
Demand	350/300	400	200	950

Destination

Source	2	3	4	Supply
2	[300] 6	5	9	400/100
3	3	3	2	500
Demand	300/0	400	200	900

Destination

		3	4	Supply
Source	2	$\boxed{100}$ 5	9	100/0
	3	3	2	500
Demand		400/300	200	600

Destination

		3	4	Supply
Source	3	$\boxed{300}$ 3	$\boxed{200}$ 2	500/200/0
Demand		300/0	200/0	500

Minimum cost $= 250 \times 3 + 1 \times 50 + 300 \times 6 + 100 \times 5 + 300 \times 3 + 200 \times 2$

$= 750 + 50 + 1800 + 500 + 900 + 400$

$= 4400$

Feasable solution –

$x_{11} = 250$ $\quad$ $x_{23} = 100$

$x_{12} = 50$ $\quad$ $x_{33} = 300$

$x_{22} = 300$ $\quad$ $x_{34} = 200$

Ques –

Wordhouse

		E	F	G	Available
	A	10	8	9	15
	B	5	2	3	20
Factory	C	6	7	4	30
	D	7	6	8	35
Demand		25	26	49	100

Sol:

Wordhouse

	E	F	G	
A	$\boxed{15}$ 10	8	9	15/0
B	$\boxed{10}$ 5	$\boxed{10}$ 2	3	20/10/0
C	6	$\boxed{16}$ 7	$\boxed{14}$ 4	30/14
D	7	6	$\boxed{35}$ 8	35/0
	25/10/0	26/16/0	49/35/0	100

min cost $= 15 \times 10 + 10 \times 5 + 10 \times 2 + 16 \times 7 + 14 \times 4 + 35 \times 8$

$= 150 + 50 + 20 + 112 + 56 + 280$

$= 668$

Feasable solution-

$x_{AE} = 15$ $\quad$ $x_{CF} = 16$

$x_{BE} = 10$ $\quad$ $x_{CG} = 14$

$x_{BF} = 10$ $\quad$ $x_{DG} = 35$

Ques-

Destination

Source	1	2	3	4	Supply
1	5	10	4	5	10
2	6	8	7	5	25
3	4	2	5	7	20
Demand	25	10	15	5	55

Sol:

Destination

Source	1	2	3	4	Supply
1	$\boxed{10}$ 5	10	4	5	10/0
2	$\boxed{15}$ 6	$\boxed{10}$ 8	7	5	25/10/0
3	4	2	$\boxed{15}$ 5	$\boxed{5}$ 7	20/5/0
Demand	25/15/0	10/0	15/0	5/0	55

Min cost $= 10 \times 5 + 15 \times 6 + 10 \times 8 + 15 \times 5 + 7 \times 5$

$= 50 + 90 + 80 + 75 + 35$

$= 330$

Feasable solution-

$x_{11} = 10$ $\quad$ $x_{33} = 15$

$x_{21} = 15$ $\quad$ $x_{34} = 5$

* Least Cost Cell Method :-

Ques - Solve by least cost cell method the following transportation problem -

	1	2	3	4	Supply
1	3	1	7	4	300
2	2	6	5	9	400
3	8	3	3	2	500
Demand	250	350	400	200	1200

Sol:

	1	2	3	4	
1	3	[300] 1	7	4	300/0
2	[250] 2	[50] 6	[100] 5	9	400/150/0
3	8	3	[300] 3	[200] 2	500/300/0
	250/0	350/50	400/100/0	200/0	1200

Minimum cost $= 300\times1 + 250\times2 + 50\times6 + 100\times5 + 300\times3 + 200\times2$

$= 300 + 500 + 300 + 500 + 900 + 400$

$= 2900$ Rs.

Initial basic feasable solution-

$x_{12} = 300$ $\quad x_{23} = 100$

$x_{21} = 250$ $\quad x_{33} = 300$

$x_{22} = 50$ $\quad x_{34} = 200$

Ques-

	D_1	D_2	D_3	D_4	
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
	5	8	7	14	34

Sol:

	D_1	D_2	D_3	D_4	Supply
Q_1	19	30	50	$\boxed{7}$ 10	7/0
Q_2	$\boxed{2}$ 70	30	$\boxed{7}$ 40	60	9/2/0
Q_3	$\boxed{3}$ 40	$\boxed{8}$ 8	70	$\boxed{7}$ 20	18/10/3/0
Demand	5/2/0	8/0	7/0	14/7/0	

$$\text{Min. cost} = 7\times10 + 2\times70 + 7\times40 + 3\times40 + 8\times8 + 7\times20$$
$$= 70 + 140 + 280 + 120 + 64 + 140$$
$$= 814 \text{ Rs.} \quad \text{Ans.}$$

★ Vogal's Approximation method (VAM method) or Penatly method –

Ques–

	D_1	D_2	D_3	D_4	Supply
Q_1	19	30	50	10	7
Q_2	70	30	40	60	9
Q_3	40	8	70	20	18
Demand	5	8	7	14	

Sol:

	D_1	D_2	D_3	D_4	Supply	Penalty
Q_1	19	30	50	10	7	9
Q_2	70	30	40	60	9	10
Q_3	40	$\boxed{8}$ 8	70	20	18	12
Demand	5	8/0	7	14		
Penalty	21	22 (max)	10	10		

	D_1	D_3	D_4	Supply	Penalty
Q_1	$\boxed{5}$ 19	50	10	7/2	9
Q_2	70	40	60	9	20
Q_3	40	70	20	10	20
Demand	5/0	7	14		
Penalty	21 (max)	7	10		

	D_3	D_4	Supply	Penalty
Q_1	50	10	2	40
Q_2	40	60	9	20
Q_3	70	$\boxed{10}$ 20	10/0	50 ← (max)
Demand	7	14/4		
Penalty	10	10		

	D_3	D_4	supply	Penalty
Q_1	50	$\boxed{2}$ 10	2/0	40
Q_2	40	60	9	20
Demand	7	4/2		
Penalty	10	50 (max)		

	D_3	D_4	Supply
Q_2	$\boxed{7}$ 40	$\boxed{2}$ 60	9/2/0
Demand	7/0	2/0	

Minimum cost $= 8 \times 8 + 5 \times 19 + 10 \times 20 + 2 \times 10 + 7 \times 40 + 2 \times 60$

$= 64 + 95 + 200 + 20 + 280 + 120$

$= 779$ Rs.

Ques – Solve by VAM method –

	D_1	D_2	D_3	supply
I	10	8	9	15
II	5	2	3	20
III	6	7	4	30
IV	7	6	8	35
Demand	25	26	49	

	D_1	D_2	D_3	Supply	Penalty
I	10	8	9	15	1
II	5	$\boxed{20}$ 2	3	20/0	1
III	6	7	4	30	2
IV	7	6	8	35	1
Demand	25	26/6	49	100	
Penalty	1	4 (max)	1		

	D_1	D_2	D_3	Supply	Penalty
I	10	8	9	15	1
III	6	7	$\boxed{30}$ 4	30/0	2
IV	7	6	8	35	1
Demand	25	6	49/19	80	
Penalty	1	2	4 (max)		

	D_1	D_2	D_3	Supply	Penalty
I	10	8	9	15	1
IV	$\boxed{25}$ 7	6	8	35/10	1
Demand	25/0	6	19	50	
Penalty	3 (max)	2	1		

	D_2	D_3	Supply	Penalty
	8	$\boxed{15}$ 9	15	1
	$\boxed{6}$ 6	$\boxed{4}$ 8	10/4/0	2
Demand	6/0	19/15	25	
Penalty	2 (max)	1		

Min cost $= 2 \times 20 + 30 \times 4 + 7 \times 25 + 6 \times 6 + 15 \times 9 + 4 \times 8$

$= 40 + 120 + 175 + 36 + 135 + 32$

$= 538$ Rs.

Feasable solution - $x_{22} = 20$ $\quad x_{13} = 15$

$x_{33} = 30$ $\quad x_{43} = 4$

$x_{41} = 25$ $\quad x_{42} = 6$

★ MODI Method :-

Step 1- Find the initial basic feasable solution by using any of three methods.

Step 2- Check the number of occupied cells. If these are less than $m+n-1$, there exists degeneracy and we introduce a very small positive assignment of $\epsilon (\epsilon \to 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$.

Step 3- For each occupied cell in the current solution, we solve the equation

$$c_{ij} = u_i + v_j$$

Starting initially with some $u_i = 0$ or $v_j = 0$.

Step 4- For all unoccupied cells

$$d_{ij} = c_{ij} - (u_i + v_j)$$

Step 5- If $d_{ij} \geq 0$ then the current basic feasable solution is an optimum one. If at least one $d_{ij} \leq 0$ select the unoccupied cell having the largest positive net evaluation to enter the basis.

Step 6- Let the unoccupied cell (r, s) enter the basis. Allocate an unknown quantity say θ to the cell (r, s). Identify a loop that starts and enters at the cell (r, s) and connects some of the basic cells. Add and subtract interchangeably θ to and from the transition cells of the loop.

Step 7 – Assign a maximum value θ in such a way that the value of one basic variable becomes zero and other basic variables remain non-negative. The basic cell whose allocation has been reduced to zero leaves the basis.

Step 8 – Return the step 3 and repeat the process until an optimum basic feasable solution has been obtained.

Ques –

19	30	50	10	7
70	30	40	60	9
70	8	70	20	10
5	8	7	14	34

$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7	u_1
70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	9	u_2
70	$\boxed{8}$ 8	70	$\boxed{10}$ 20	10	u_3
5	8	7	14	34	
v_1	v_2	v_3	v_4		

$m+n-1 = 3+4-1 = 6 =$ Occupied cell

Calculate for occupied cells –

$C_{ij} = u_i + v_j$

$C_{11} = u_1 + v_1 \Rightarrow 19 = u_1 + v_1$

$C_{14} = u_1 + v_4 \Rightarrow 10 = u_1 + v_4$

$C_{23} = u_2 + v_3 \Rightarrow 40 = u_2 + v_3$

$C_{24} = u_2 + v_4 \Rightarrow 60 = u_2 + v_4$

$C_{32} = u_3 + v_2 \Rightarrow 8 = u_3 + v_2$

$C_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + v_4$

Put $v_4 = 0$

$u_3 = 20$	$v_1 = 9$
$u_2 = 60$	$v_3 = -20$
$u_1 = 10$	$v_2 = -12$

Calculate for non- occupied cells-

$d_{ij} = C_{ij} - (u_i + v_j) \Rightarrow$

$d_{12} = 30 - (u_1 + v_2) \Rightarrow 30 - (10 - 12) = 32$

$d_{13} = 50 - (u_1 + v_3) \Rightarrow 50 - (10 - 20) = 60$

$d_{21} = 70 - (u_2 + v_1) \Rightarrow 70 - (60 + 9) = 1$

$d_{22} = 30 - (u_2 + v_2) \Rightarrow 30 - (60 - 12) = -18$

$d_{31} = 70 - (u_3 + v_1) \Rightarrow 70 - (20 + 9) = 41$

$d_{33} = 70 - (u_3 + v_3) \Rightarrow 70 - (20 - 20) = 70$

$d_{22} \le 0$, This is not optimum

5			2
	$\boxed{2}$ +	7	− 2 $\boxed{10}$
	$\boxed{6}$ 8 −		+ 10 $\boxed{12}$

Add 2 to (+ve) sign and subtract at (−ve) sign

$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7	u_1
70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	9	u_2
70	$\boxed{6}$ 8	70	$\boxed{12}$ 20	18	u_3
5	8	7	14	34	
v_1	v_2	v_3	v_4		

Calculate by occupied cells-

$C_{ij} = u_i + v_j$

$C_{11} = u_1 + v_1 \Rightarrow 19 = u_1 + v_1$

$C_{14} = u_1 + v_4 \Rightarrow 10 = u_1 + v_4$

$C_{22} = u_2 + v_2 \Rightarrow 30 = u_2 + v_2$

$C_{23} = u_2 + v_3 \Rightarrow 40 = u_2 + v_3$

$C_{32} = u_3 + v_2 \Rightarrow 8 = u_3 + v_2$

$C_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + v_4$

Put $u_1 = 0$

$v_1 = 19$	$u_2 = 32$
$v_4 = 10$	$v_3 = 8$
$v_2 = -2$	$u_3 = 10$

Calculate for non-occupied cells-

$d_{ij} = C_{ij} - (u_i + v_j)$

$d_{12} = 30 - (u_1 + v_2) \Rightarrow 30 - (0 - 2) = 32$

$d_{13} = 50 - (u_1 + v_3) \Rightarrow 50 - (0 + 8) = 42$

$d_{21} = 70 - (u_2 + v_1) \Rightarrow 70 - (32 + 9) = 19$

$d_{24} = 60 - (u_2 + v_4) \Rightarrow 60 - (32 + 10) = 18$

$d_{31} = 70 - (u_3 + v_1) \Rightarrow 70 - (10 + 19) = 41$

$d_{33} = 70 - (u_3 + v_3) \Rightarrow 70 - (10 + 8) = 52$

$d_{ij} \geq 0$ optimum solution

New initial basic feasable solution

$x_{11} = 5$ $\quad x_{23} = 7$

$x_{14} = 2$ $\quad x_{32} = 6$

$x_{22} = 2$ $\quad x_{34} = 12$

Ques- Find the optimal solution of transportation problem-

	A	B	C	Available
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	8

	A	B	C	Available	Penalty
I	50	30	220	1	20
II	90	45	170	3	45
III	250	200	2 50	4/2	150 (max)
Requirement	4	2	2/0	8	
Penalty	40	15	120		

	A	B	Available	Penalty
I	50	30	1	20
II	90	45	3	45
III	250	2 200	2/0	50(max)
Requirement	4	2/0	6	
Penalty	40	15		

	A	Available
I	1 50	1/0
II	3 90	3/0
	4/3/0	4

	A	B	C	
I	$\boxed{1}$ 50	30	220	1
II	$\boxed{3}$ 90	45	170	3
III	250	$\boxed{2}$ 200	$\boxed{2}$ 50	4
	4	2	2	8

$$m + n - 1 = 3 + 3 - 1 = 5$$

* Assignment problem :-

The assignment problem can be stated in the form of $(n \times n)$ square cost matrix of real number. Suppose there are n jobs and n persons are available for doing these jobs. Assume that each person can do each job at a time.

Let c_{ij} is the cost if the i^{th} person is assign the j^{th} job. The problem is to find an assignment that is which job should be assign to which person, so that the Total cost for performing all jobs is minimum.

$$\begin{array}{cc|ccccccc|}
 & & \multicolumn{7}{c}{\text{Jobs}} \\
 & & 1 & 2 & 3 & \cdots & j & \cdots & n \\
\hline
 & 1 & C_{11} & C_{12} & C_{13} & \cdots & C_{1j} & \cdots & C_{1n} \\
 & 2 & C_{21} & C_{22} & C_{23} & \cdots & C_{2j} & \cdots & C_{2n} \\
\text{Persons} & 3 & C_{31} & C_{32} & C_{33} & \cdots & C_{3j} & \cdots & C_{3n} \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
 & i & C_{i1} & C_{i2} & C_{i3} & \cdots & C_{ij} & \cdots & C_{in} \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
 & n & C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nj} & \cdots & C_{nn} \\
\hline
\end{array}$$

Mathematical formulation of Assignment Problem -

Mathematically, the assignment problem can be stated as -

The total minimum cost

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

where $i = 1, 2, 3 \dots n$
$j = 1, 2, 3 \dots n$

Subject to restrictions

$$x_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ person assigned } j^{th} \text{ job.} \\ 0, & \text{if not.} \end{cases}$$

$\sum_{j=1}^{n} x_{ij} = 1$ (one job is done by the i^{th} person, $i=1,2,3 \ldots n$)

$\sum_{i=1}^{n} x_{ij} = 1$ (only one person should be assigned the j^{th} job, $j = 1,2,3 \ldots n$)

where x_{ij} denotes the j^{th} job is to be assigned to the i^{th} person.

* Difference between Assignment and Transportation Problem–

1- Assignment is the square matrix but transportation is not necessary square matrix if we put $m=n$ in transportation then it becomes the problem of assignment. Hence we say that assignment problem is the special case of transportation problem.

2- The numerical evaluations of such association are called 'effectiveness' instead of 'transportation costs'.

3- Mathematically, all a_i and b_j are unity and each x_{ij} is limited to one of the two values 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero. One of each origin and one for each destination.

Ques–

Jobs

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Sol:

	1	2	3	4	5
A	7	3	1	5	0
B	0	9	5	5	4
C	1	6	7	0	4
D	4	3	1	0	3
E	4	0	3	4	0

	1	2	3	4	5
A	7	3	$\cancel{0}$	5	$\boxed{0}$
B	$\boxed{0}$	9	4	5	4
C	1	6	6	$\boxed{0}$	4
D	4	3	$\boxed{0}$	$\cancel{0}$	3
E	4	$\boxed{0}$	2	4	$\cancel{0}$

$A \to 5$, $B \to 1$, $C \to 4$, $D \to 3$, $E \to 2$

Minimum cost $= 1+0+2+1+5$

$= 9$ Ans.

Ques- Solve the assignment problem –

	I	II	III	IV
A	2	3	4	5
B	4	5	6	7
C	7	8	9	8
D	3	5	8	4

Sol:- Step 1- Subtract the smallest element of each row from every element of the corresponding row in the given matrix.

	I	II	III	IV
A	0	1	2	3
B	0	1	2	3
C	0	1	2	1
D	0	2	5	1

Step 2– Subtract the smallest element of each column from the every element of the corresponding column in the given matrix.

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 0 & 0 & 0 & 2 \\ B & 0 & 0 & 0 & 2 \\ C & 0 & 0 & 0 & 0 \\ D & 0 & 1 & 3 & 0 \\ \hline \end{array}$$

Step 3– Make the assignment.

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & \boxed{0} & \cancel{0} & \cancel{0} & 2 \\ B & \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ C & \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0} \\ D & \cancel{0} & 1 & 3 & \boxed{0} \\ \hline \end{array}$$

$A \to I$, $B \to II$, $C \to III$, $D \to IV$

Total minimum cost $= 2+5+9+4$

$= 20$ Ans.

Ques– Solve the assignment problem–

$$\begin{array}{c|ccc|} & M_1 & M_2 & M_3 \\ \hline J_1 & 8 & 7 & 6 \\ J_2 & 5 & 7 & 8 \\ J_3 & 6 & 8 & 7 \\ \hline \end{array}$$

Sol:- Step 1- Subtract the smallest element of each row from every element of the corresponding row in the given matrix.

$$\begin{array}{c|ccc|} & M_1 & M_2 & M_3 \\ \hline J_1 & 2 & 1 & 0 \\ J_2 & 0 & 2 & 3 \\ J_3 & 0 & 2 & 1 \\ \hline \end{array}$$

	I	II	III	IV	V	
A	30	$\boxed{0}$	35	30	15	✓
B	15	$\cancel{0}$	$\boxed{\cancel{0}}$	10	$\cancel{0}$	
C	30	$\cancel{0}$	35	30	20	✓
D	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
E	20	$\cancel{0}$	25	15	15	✓
		✓				

	I	II	III	IV	V
A	15	$\cancel{0}$	20	15	$\boxed{0}$
B	15	15	$\boxed{0}$	10	$\cancel{0}$
C	15	$\boxed{0}$	20	15	5
D	$\boxed{0}$	15	20	$\cancel{0}$	5
E	5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

~~Hence~~ $A \to V,\ B \to III,\ C \to II,\ D \to I,\ E \to IV$

Minimum cost $= 200 + 130 + 110 + 50 + 80$

$= 570$ Rs. Ans.

Ques Solve the assignment problem—

	I	II	III	IV
a	18	24	28	32
b	8	13	17	19
c	10	15	19	22

Sol: Given problem is an unbalanced assignment problem so we add a dummy row.

	I	II	III	IV
a	18	24	28	32
b	8	13	17	19
c	10	15	19	22
d	0	0	0	0

Step 2—

	I	II	III	IV
a	0	6	10	14
b	0	5	9	11
c	0	5	9	12
d	0	0	0	0

Step 3—

	I	II	III	IV	
a	$\boxed{0}$	6	10	14	✓
b	$\cancel{0}$	5	9	11	✓
c	$\cancel{0}$	5	9	12	✓
d	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
	✓				

Step 4 –

	I	II	III	IV	
a	$\boxed{0}$	1	5	9	✓
b	$\cancel{0}$	$\boxed{0}$	4	6	✓
c	$\cancel{0}$	$\cancel{0}$	4	7	✓
d	5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	
	✓	✓			

Step 5 –

	I	II	III	IV
a	$\boxed{0}$	1	1	5
b	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
c	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
d	9	4	$\cancel{0}$	$\boxed{0}$

$a \to I,\ b \to II,\ c \to III,\ d - IV$

Minimum cost = 18 + 13 + 19 + 0

= 50 Rs. Ans.

Ques – Solve the assignment problem –

5	3	1	8
7	9	2	6
6	4	5	7
5	7	7	6

Sol: Step 1 –

4	2	0	7
5	7	0	4
2	0	1	3
0	2	2	1

Step 2 –

4	2	0	6
5	7	0	3
2	0	1	2
0	2	2	0

Step 3 –

4	2	$\boxed{0}$	6	✓
5	7	$\cancel{0}$	3	✓
2	$\boxed{0}$	1	2	
$\boxed{0}$	2	2	$\cancel{0}$	
		✓		

Step 4 –

2	$\boxed{0}$	$\cancel{0}$	4	✓
3	5	$\boxed{0}$	1	
2	$\cancel{0}$	3	2	✓
$\boxed{0}$	2	4	$\cancel{0}$	
	✓			

$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	3
3	7	$\boxed{0}$	1
$\boxed{0}$	$\cancel{0}$	1	$\cancel{0}$
$\cancel{0}$	4	4	$\boxed{0}$

Min cost = 3 + 2 + 6 + 6

= 17 Ans.

6 Mar 2019

UNIT - 2

Queuing Theory

* Queuing theory :- A group of items waiting to receive services, including those receiving the service is known as waiting line or a queue.
  Waiting line or queue are omnipresent. Business of all types, industry, school, hospitals, school, hospitals, book stalls, banks, post office, petrol pump, all have queuing problems.

* The basic queuing process and its characteristics - The basic queuing process can be described as a process in which the customers arrive for service at a service counter or station, wait for their turn in the queue if the server is busy in the service of the customer and are served when the server gets free and the customer leave the system as soon as he is served.

Characteristics of Queuing System -

1- Arrival distribution - It represents the pattern in which the number of customers arrive at the system. Arrivals may also be represented by the inter-arrival time, which is the period between two successive arrivals.
The rate at which customers arrive to be serviced, i.e. number of customers arriving per unit of time is called arrival rate. When the arrival rate is random, the customers arrive in no logical pattern.

This represents most cases in the business world.

2- Service distribution - It represents the pattern in which the number of customers leave the system. Departures may also be represented by the service time which is the time between two successive services.

3- Service channels - The queuing system may have a single service channel. Arriving customers may form one line and get serviced, as in a doctor's clinic. In case of parallel channels, several customers may be serviced simultaneously as in a barber shop. A queuing model is called one service model when the system has one server only and a multi-server channel (model) when the system has a number of parallel channels each with one server.

4- Service discipline - The service discipline refers to the manner in which the members in the queue are chosen for service. The following service disciplines are seen in common practice.

(i) First Come, First Serve (FCFS) - According to this discipline the customers are served in the order of their arrival. This service discipline may be seen at a railway ticket window etc.

(ii) Last Come, First Serve (LCFS) - According to this discipline the items arriving last are taken out first. This discipline may be seen in godowns where the units (items) which come last are taken out first.

(iii) Service in random order.

(iv) Service on some Priority - Procedure — Some customers are served before the others without considering their order of arrival.

5- Maximum number of Customers allowed in the system — The Maximum numbers of customers in the system can be either finite or infinite. In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system.

6- Service Mechanism — The service mechanism refers to

(i) the pattern according to which the customers are served.

(ii) facilities given to the customers:

(a) Single-channel: Here the customers are served by one counter only.

(b) Multi-channel: Here the customers are served by more than one counter.

★ Definitions in Queuing Problem :-

1- Queue length — Queue length is defined by the number of persons (customers) waiting in the line at any time.

2- Average length of line – Average length of line is defined by the number of customers in the queue per unit time.

3- Waiting time – It is time upto which a unit has to wait in the queue before it is taken into service.

4- Servicing time – The time taken for servicing of a unit is called its servicing time.

5- Idle period – When all the units in the queue are served. The idle period of the server begins and it continuous upto the time of arrival of the unit. The idle period of a server is the time during which he remains free because there is no customers present in the system.

6- Mean arrival rate – The mean arrival rate in a waiting-line situation is defined as the expected number of arrivals occuring in a time interval of length unity.

7- Mean servicing rate – The mean servicing rate for a particular servicing station is defined as the expected number of services completed in a time interval of length unity, given that the servicing is going on throughout the entire time unit.

8- Steady state – A system is said to be in steady state when its operating characteristics becomes independent of time.

9- Traffic intensity – In case of a simple queue the traffic intensity is the ratio of mean arrival rate and the mean servicing rate.

$$\text{Traffic intensity} = \frac{\text{Mean arrival rate}}{\text{Mean servicing rate}}$$

* Poisson Process :-

Theorem :- In Poisson process the probability of n arrivals during time intervals of length t is given by -

$$P_n(\Delta t) = \frac{(\lambda \Delta t)^n e^{-\lambda \Delta t}}{\lfloor n}$$

where λ, t is the parameter.

Proof :-

Case I – $n = 0$ (no arrival)

$$P_0(\Delta t) = \frac{(\lambda \Delta t)^0 e^{-\lambda \Delta t}}{\lfloor 0}$$

$$P_0(\Delta t) = e^{-\lambda \Delta t}$$

$$P_0(\Delta t) = 1 - \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} - \frac{(\lambda \Delta t)^3}{\lfloor 3} \quad \left[e^{-x} = 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} \ldots\right]$$

$o(\Delta t) \to 0$ when Δt is very small

$$\boxed{P_0(\Delta t) = 1 - \lambda \Delta t}$$

Case II – $n = 1$

$$P_1(\Delta t) = \frac{(\lambda\Delta t)^1 e^{-\lambda\Delta t}}{\lfloor 1}$$

$$= \lambda\Delta t \left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{\lfloor 2} + \dots\right]$$

$$= \lambda\Delta t - (\lambda\Delta t)^2 + \frac{(\lambda\Delta t)^3}{\lfloor 2} + \dots$$

$O(\Delta t) \to 0$

$$\boxed{P_1(\Delta t) = \lambda\Delta t}$$

Case III – $n = m > 1$ (m is more than one)

$$P_m(\Delta t) = \frac{(\lambda\Delta t)^m e^{-\lambda\Delta t}}{\lfloor m}$$

$$P_m(\Delta t) = \frac{(\lambda\Delta t)^m}{\lfloor m}\left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{\lfloor 2} - \frac{(\lambda\Delta t)^3}{\lfloor 3} + \dots\right]$$

$\Delta t \to 0$

$$\boxed{P_m(\Delta t) = 0}$$

$$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow P_0 + P_1 + P_2 + \dots$$

$$= 1 - \lambda\cancel{\Delta t} + \lambda\cancel{\Delta t} + \dots$$

$$\boxed{\sum_{n=0}^{\infty} P_n = 1}$$

★ **Model I (M/M/1) : (∞/FCFS) (Birth and death model) :-**

This is the queuing model with poisson arrival, poisson service, single channel with infinite capacity. The service discipline is first come first service. Here λ is the mean arrival rate and μ is the mean service rate.

To find the steady state equation –

t	arrival capacity (λ) (Δt)	Service capacity (μ) (Δt)	$t+\Delta t$
n	1	1	n
$n+1$	0	1	n
$n-1$	1	0	n
n	0	0	n

$$P_n(t+\Delta t) = P(\text{case I}) + P(\text{case II}) + P(\text{case III}) + P(\text{case IV})$$

$$P_n(t+\Delta t) = P_n(t)(\lambda\Delta t)(\mu\Delta t) + P_{n+1}(t)(1-\lambda\Delta t)(\mu\Delta t) + P_{n-1}(t)\lambda\Delta t(1-\mu\Delta t) + P_n(t)(1-\lambda\Delta t)(1-\mu\Delta t)$$

$$P_n(t+\Delta t) = P_n(t)\lambda\mu(\Delta t)^2 + P_{n+1}(t)(\mu-\lambda\mu\Delta t)\Delta t + P_{n-1}(t)(\lambda-\lambda\mu\Delta t)\Delta t + P_n(t)\left[1-\lambda\Delta t-\mu\Delta t+\lambda\mu(\Delta t)^2\right]$$

$$P_n(t+\Delta t) = P_n(t)\lambda\mu(\Delta t)^2 + \mu P_{n+1}(t)\Delta t - P_{n+1}(t)\lambda\mu(\Delta t)^2 + P_{n-1}(t)\lambda\Delta t - P_{n-1}(t)\lambda\mu(\Delta t)^2 + P_n(t)\left[1-\mu\Delta t-\lambda\Delta t+\lambda\mu(\Delta t)^2\right]$$

$$P_n(t+\Delta t) = P_n(t)\lambda\mu(\Delta t)^2 + P_{n+1}(t)\mu\Delta t - P_{n+1}(t)\mu\Delta t - P_{n+1}(t)\lambda\mu(\Delta t)^2 + P_{n-1}(t)\lambda\Delta t - P_{n-1}(t)\lambda\mu(\Delta t)^2 + P_n(t) - P_n(t)\mu\Delta t - P_n(t)\lambda\Delta t + P_n(t)\lambda\mu(\Delta t)^2$$

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = P_n(t)\lambda\mu\Delta t + P_{n+1}(t)\mu - P_{n+1}(t)\lambda\mu\Delta t + P_{n-1}(t)\lambda - P_{n-1}(t)\lambda\mu\Delta t - P_n(t)\mu - P_n(t)\lambda + P_n(t)\lambda\mu\Delta t$$

$\Delta t \to 0$ (directional derivative zero)

$$0 = P_{n+1}(t)\mu + P_{n-1}(t)\lambda - P_n(t)\mu - P_n(t)\lambda$$

$$\boxed{P_n(t)(\lambda+\mu) = P_{n+1}(t)\mu + P_{n-1}(t)\lambda}$$

This is steady state equation.

1- Probability of no customers in the system is –

$$P_0 = 1 - \rho \qquad \text{where } \rho = \frac{\lambda}{\mu}$$

2- Probability of n customers in the system –

$$P_n = (1-\rho)\,\rho^n$$

3- Probability of more than n customers in the system

$$P(>n) = \rho^{n+1}$$

4- Probability of more than n customers in the queue.

$$P(>n+1) = \rho^{n+2}$$

5- The average (expected) number of customers in the system

$$L \text{ or } L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

6- Average (expected) queue length is

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}$$

7- Average (expected) waiting time of customer in the queue

$$W_q = E(W) = \frac{\lambda}{\mu(\mu-\lambda)}$$

8- Average (expected) waiting time that a customer spend in the system is

$$W_s = \frac{1}{\mu-\lambda}$$

9- The variance of queue length

$$Var(n) = \frac{\rho}{(1-\rho)^2}$$

Inter-relationship between L_s, L_q, W_s, W_q -

$$L_s = \lambda W_s$$

$$L_q = \lambda W_q$$

$$W_q = W_s - \frac{1}{\mu}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

Ques- Customers arrive at a sales counter managed by single person according to a poisson process with a mean rate of 20 per hour. The time require to serve a customer has an exponential distribution with a mean of 100 second. Find the average time of a customer.

Sol:-

$$\frac{1}{\mu} = \frac{10\cancel{0}}{6\cancel{0} \times 6\cancel{0}} = \frac{1}{36}$$

$$\mu = 36 \text{ / hour}$$

$$\lambda = 20 \text{ / hour}$$

Average waiting time in the queue -

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$= \frac{20}{36(36-20)} = \frac{20}{36 \times 16} = \frac{5}{144} \times 60 \times 60$$

$$= 125 \text{ / hr}$$

Average waiting time in the system -

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{36-20}$$

$$= \frac{1}{16} \times 60 \times 60 = 225 \text{ / hr.}$$

Ay.

★ Model II (M/M/1):(M/FCFS) Finite queue length model –

1- The probability of no customer in the system

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}, \quad \text{where } \rho = \frac{\lambda}{\mu}$$

$\lambda \rightarrow$ mean arrival rate

$\mu \rightarrow$ mean service rate

2- The probability of n customers in the system

$$P_n = \frac{1-\rho}{1-\rho^{N+1}} \rho^n$$

3- Average number of customers in the ~~queue~~ system

$$L_s = \frac{\rho\left[1-(1+N)\rho^N + N\rho^{N-1}\right]}{(1-\rho)(1-\rho^{N+1})} \quad \text{where } \rho = \frac{\lambda}{\mu}$$

4- Average number of customers in the queue

$$L_q = \frac{1-N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \rho^2 = \sum_{n=0}^{N} nP_n$$

5- Average time a customer spend in the system

$$W_s = \frac{L_s}{\lambda'} \quad \text{where } \lambda' = \lambda(1-\rho^N)$$

6- Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda'} \quad \text{where } \lambda' = \lambda(1-\rho^N)$$

15 Mar 2019

UNIT – 3
Replacement Theory

★ **Replacement Problem :-** The efficiency of all industrial and military equipments deteriorate (decreases) with time. Sometimes the equipment fails completely and effect the whole system.
For example, a machine requires higher operating cost, a transport vehicle such as car or airplane requires more and more maintenance cost. The ever increasing repair and maintenance cost necessitates the replacement of equipment.
The replacement problem are concerned with situation that arise when some items such as men, car, truck, airplane etc need replacement due to their decrease efficiency or working capacity.
There are 4 types of replacement problem (Important replacement situations) –

1- Replacement of old items has become in bad condition and their working efficiency decrease with time or require expensive maintenance.

2- Replacement of items when the system is completely fail due to accident or otherwise.

3- Problem in mortality or staffing.

4- Replacement of an equipment, when a better or more efficient design of machine or equipment has become available in the market.

For example - An equipment may have an economic life of 20 years, yet it may become obsolete after 10 years because of better technical development.

Note -

C = Cost of machine

S = Scrap value (resale value)

$C_m(t)$ = Maintenance cost of the machine at the time.

$A(n)$ = Average annual cost of the item.

n = Number of years the item is to be in use

Annual cost of the item at any time

$= C - S +$ Maintenance cost at time t

$= C - S + C_m(t)$

Ques - The cost of a machine is 6100 Rs. The scrap value (resale value) is only 100 Rs. The maintenance cost are found from experience to be under

Year	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	100	250	400	600	900	1250	1600	2000

Sol: $C = 6100$ Rs. $S = 100$ Rs.

Year(n)	Maintenance Cost $C_m(t)$	Total cost of maintenance $\sum C_m(t)$	C − S	Total cost $= C-S+\sum C_m(t)$	Average cost
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837.5
5	900	2250	6000	8250	1650 min
6	1250	3500	6000	9500	1583.3
7	1600	5100	6000	11100	1585.7
8	2000	7100	6000	13100	1637.5

Hence, the machine should be replaced after every 6th year.

Ques-

Year	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	100	250	400	600	900	1200	1600	2000

Sol:- $C = 6100$ ₹ $S = 100$ ₹

Year (n) (1)	Maintenance Cost $C_m(t)$ (2)	Total cost of maintenance $\sum C_m(t)$ (3)	C − S (4)	Total cost $= C-S+\sum C_m(t)$ (5)	Average cost $= \left(\frac{5}{n}\right)$ (6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
6	1200	3450	6000	9450	1575 ← min
7	1600	5050	6000	11050	1579
8	2000	7050	6000	13050	1631

Hence the machine should be replaced after every 6^{th} year.

Ques- Following table give the running cost and resale price of a certain equipment whose purchase price is Rs. 5000.

Year	1	2	3	4	5	6	7	8
Running cost (Rs)	1500	1600	1800	2100	2500	2900	3400	4000
Resale value (Rs.)	3500	2500	1700	1200	800	500	500	500

At what year is the replacement due?

Sol:- $C = Rs. 5000$

Year (n) (1)	Maintenance cost $C_m(t)$ (2)	Total maintenance cost $\Sigma C_m(t)$ (3)	Scrap Value S (4)	C−S (5)	Total cost $C-S+\Sigma C_m(t)$ (6)	Total average cost $= \frac{(6)}{n}$ (7)
1	1500	1500	3500	1500	3000	3000
2	1600	3100	2500	2500	5600	2800
3	1800	4900	1700	3300	8200	2733.3
4	2100	7000	1200	3800	10800	2700 (minimum)
5	2500	9500	800	4200	13700	2740
6	2900	12400	500	4500	16900	2816.6
7	3400	15800	500	4500	20300	2900
8	4000	19800	500	4500	24300	3037.5

Hence, the equipment should be replaced every 4^{th} year.

Ques- Fleet cars have increased their costs as they continue in service due to increased direct operating cost and increased maintenance. The initial cost is Rs. 3500 and trade in value drops at time passes until it reaches a constant value of Rs. 500. Given cost of trade in value determine the proper length of service before cars should be replaced.

Years of service	1	2	3	4	5
Year end trade in value(S)	1900	1050	600	500	500
Annual operating cost	1500	1800	2100	2400	2700
Annual maintaining cost	300	400	600	800	1000

Sol:- $C =$ Rs 3500

Year (n)	Running Cost	Total running cost $\Sigma C_m(t)$	C − S	Total cost $= C - S + \Sigma C_m(t)$	Average cost
1	1800	1800	1600	3400	3400
2	2200	4000	2450	6450	3225
3	2700	6700	2900	9600	3200 (minimum)
4	3200	9900	3000	12900	3225
5	3700	13600	3000	16600	3320

Hence, the car should be replaced every 3rd year.

Ques- The cost of a truck is 10,000 Rs. The salvage value and the running cost are given below. Find the most economical age for replacement?

Year	1	2	3	4	5	6	7
Running cost (Rs.)	3000	3200	3600	4200	5000	5800	6800
Resale value (Rs)	7000	5000	3400	2400	1600	1600	1000

Sol:- $C =$ 10,000 Rs.

Year (n)	Running Cost $C_m(t)$	Total running Cost $\Sigma C_m(t)$	Scrap Value S	C − S	Total cost $= C-S+\Sigma C_m(t)$	Average cost
1	3000	3000	7000	3000	6000	6000
2	3200	6200	5000	5000	11200	5600
3	3600	9800	3400	6600	16400	5466.6
4	4200	14000	2400	7600	21600	5400 (minimum)
5	5000	19000	1600	8400	27400	5480
6	5800	24800	1600	8400	33200	5533.3
7	6800	31600	1000	9000	40600	5800

Hence, the truck should be replaced every 4^{th} year.

Ques- A firm is considering when to replace its machine whose price is 12,200 Rs. The scrap value of machine is 200 Rs. only. From past experience the maintenance cost of the machine are as under –

Year	1	2	3	4	5	6	7	8
Running Cost (Rs.)	200	500	800	1200	1800	2500	3200	4000

Find when the new machine should be purchased?

Sol:- $C =$ Rs. 12,200 $S =$ Rs. 200

Year (n)	Running Cost $C_m(t)$	Total Running Cost $\Sigma C_m(t)$	Scrap Value S	C − S	Total cost $= C-S+\Sigma C_m(t)$	Average Cost
1	200	200	200	12000	12200	12200
2	500	700	200	12000	12700	6350
3	800	1500	200	12000	13500	4500
4	1200	2700	200	12000	14700	3675
5	1800	4500	200	12000	16500	3300
6	2500	7000	200	12000	19000	3166.6 min
7	3200	10200	200	12000	22200	3171.4
8	4000	14200	200	12000	26200	3275

Hence, the machine should be replaced every 6^{th} year.

Ques- The following mortality rates have been observed for a certain type of bulb.

Weekly	1	2	3	4	5
Percent failing by end of the week	10	25	50	80	100

There are 1000 bulbs in use and it costs Rs. 1.00 to replace an individual bulb each has burst out. If all bulbs were replaced simultaneously it would cost 25 paise per bulb. It is purposed to replace all bulbs at fixed intervals, whether or not they have burnt out and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced?

Sol:- Let P_i be the prob. of light bulb, which was now placed in position for use, fail during the ith week of its life.

$P_1 \Rightarrow$ The Prob of failure in the 1st week $= \frac{10}{100} = 0.10$

$P_2 \Rightarrow$ The Prob of failure in the 2nd week $= \frac{25-10}{100} = 0.15$

$P_3 \Rightarrow$ The Prob of failure in the 3rd week $= \frac{50-25}{100} = 0.25$

$P_4 \Rightarrow$ The Prob of failure in the 4th week $= \frac{80-50}{100} = 0.30$

$P_5 \Rightarrow$ The Prob of failure in the 5th week $= \frac{100-80}{100} = 0.20$

Here the Probability $= P_1 + P_2 + P_3 + P_4 + P_5$

$= 0.10 + 0.15 + 0.25 + 0.30 + 0.20$

$= 1$

and $P_6 + P_7 + P_8 + \ldots$

$= 0$

Let N_i be the number of bulb in the beginning

$N_0 = 1000$

$N_1 \Rightarrow$ No. of burnt out bulbs replaced in the end of 1st week

$N_1 = N_0 P_1$

$= 1000 \times 0.10 = 100$

$N_2 \Rightarrow$ No. of burnt out bulbs replaced in the end of 2nd week.

$N_2 = N_0 P_2 + N_1 P_1$

$= 1000 \times 0.15 + 100 \times 0.10$

$= 150 + 10 = 160$

$N_3 \Rightarrow$ No. of burnt out bulbs replaced in the end of 3rd week.

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 1000 \times 0.25 + 100 \times 0.15 + 160 \times 0.10$

$= 250 + 15 + 16 = 281$

$N_4 \Rightarrow$ No. of burnt out bulbs replaced in the end of 4th week.

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 1000 \times 0.30 + 100 \times 0.25 + 160 \times 0.15 + 281 \times 0.10$

$= 300 + 25 + 24 + 28.1 = 377$

$N_5 \Rightarrow$ No. of burnt out bulbs replaced in the end of 5th week.

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 1000 \times 0.20 + 100 \times 0.30 + 160 \times 0.25 + 281 \times 0.15$
$+ 377 \times 0.10$

$= 200 + 30 + 40 + 42.15 + 37.7$

$= 350$

$N_6 = N_0 P_6 + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$

$= 0 + 230$

$= 230$

$N_7 = 286$

$N_8 = 320$

Average no. of replacement per week

$= \frac{No.}{Mean\ age} = \frac{1000}{3.35} = 299$

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Average life of the bulbs $= \sum X_i P_i$

$X_i \rightarrow$ Week , $P_i \rightarrow$ Probability

$\Rightarrow X_1P_1 + X_2P_2 + X_3P_3 + X_4P_4 + X_5P_5 + X_6P_6$

$= 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 + 6P_6$

$= 0.10 + 2 \times 0.15 + 3 \times 0.25 + 4 \times 0.30 + 5 \times 0.20 + 0$

$= 0.10 + 0.30 + 0.75 + 1.2 + 1$

$= 3.35$

End of the week	Total cost of group replacement	Average cost per week (Rs)
1	$1000 \times 0.25 + 100 \times 1 = 350$	350
2	$1000 \times 0.25 + (100+160) = 510$	255 ← min
3	$1000 \times 0.25 + (100+160+281) = 791$	263
4	$1000 \times 0.25 + (160+160+281+377) = 1168$	292
5	$1000 \times 0.25 + (100+160+281+377+350) = 1518$	303.6

We replace bulbs after every 2nd week.

Ques- It has been suggested by a data processing firm that these adopt a policy of periodically replacing all the tubes in a certain piece of equipment. A given type of tube is known to have the mortality distribution shown in the table -

Tube failure/week	1	2	3	4	5
Probability of failure	0.3	0.1	0.1	0.2	0.3

There are approximately 1000 tubes of this type in all the combined equipment. The cost of replacing the tubes on an individual basis is estimated to be 1 Rs. per tube and the cost of a group replacement policy average 0.30 Rs. per tube. Compare the cost of preventive replacement with that of remedial replacement.

Sol:- Let N_i be the number of tubes in the beginning.

$N_0 = 1000$

$N_1 \Rightarrow$ Tubes replaced in the end of 1st week

$N_1 = N_0 P_1$

$= 1000 \times 0.3 = 300$

$N_2 \Rightarrow$ Tubes replaced in the end of 2nd week

$N_2 = N_0 P_2 + N_1 P_1$

$= 1000 \times 0.1 + 300 \times 0.3$

$= 100 + 90 = 190$

$N_3 \Rightarrow$ Tubes replaced in the end of 3rd week

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 1000 \times 0.1 + 300 \times 0.1 + 190 \times 0.3$

$= 100 + 30 + 57 = 187$

$N_4 \Rightarrow$ Tubes replaced in the end of 4th week

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 1000 \times 0.2 + 300 \times 0.1 + 190 \times 0.1 + 187 \times 0.3$

$= 200 + 30 + 19 + 56.1$

$= 305$

$N_5 \Rightarrow$ Tubes replaced in the end of 5th week

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 1000 \times 0.3 + 300 \times 0.2 + 190 \times 0.1 + 187 \times 0.1 + 305 \times 0.3$

$= 300 + 60 + 19 + 18.7 + 91.5$

$= 489.2$

Average life of the tubes $= \Sigma X_i P_i$

$X_i \rightarrow$ Week , $P_i \rightarrow$ Probability

$\Rightarrow X_1 P_1 + X_2 P_2 + X_3 P_3 + X_4 P_4 + X_5 P_5 + X_6 P_6$

$= 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_6 + 0$

$= 0.3 + 2 \times 0.1 + 3 \times 0.1 + 4 \times 0.2 + 5 \times 0.3$

$= 0.3 + 0.2 + 0.3 + 0.8 + 1.5$

$= 3.1$

Average no. of replacements per week = $\frac{\text{No.}}{\text{Mean age}}$

$= \frac{1000}{3.1} = 322$

End of the week	Total cost of group replacement	Average cost per week (Rs.)
1	$1000 \times 0.30 + 300 \times 1 = 600$	600
2	$300 + (300 + 190) = 790$	395
3	$300 + (300 + 190 + 187) = 977$	325.6
4	$300 + (300 + 190 + 187 + 305) = 1282$	320.5 ← min
5	$300 + (300 + 190 + 187 + 305) = 1771$	354.2

The tubes should replaced at every 4^{th} week.

ues – The following mortality rates have been observed for a certain type of light bulbs:

End of week	1	2	3	4	5	6
Prob. of failure to date	0.09	0.25	0.49	0.85	0.97	1.00

There are large number of such bulbs which are to be kept in working order. If bulb fail in service it cost Rs. 3 to replace but if all the bulbs are replaced in the same operation. It can be done for only Rs 0.70 It is proposed to replace all bulbs at fixed intervals whether or not they have burnt out and to continue replacing burnts as they fail.

(i) What is the best interval between group replacements

(ii) At what group replacement price per would a policy of strictly individual replacement become preferable to the adopted policy?

Sol:- Let P_i be the Prob. of light bulbs, which was now placed in position for use, fail during the ith week of its life.

$P_1 \Rightarrow$ The Prob. of failure in the 1^{st} week $= 0.09$

$P_2 \Rightarrow$ " " " " " 2^{nd} week $= 0.25 - 0.09$

$P_2 = 0.16$

$P_3 \Rightarrow$ " " " " " 3^{rd} week $= 0.49 - 0.25$

$P_3 = 0.24$

$P_4 \Rightarrow$ " " " " " 4^{th} week $= 0.85 - 0.49$

$P_4 = 0.36$

$P_5 \Rightarrow$ " " " " " 5^{th} week $= 0.97 - 0.85$

$P_5 = 0.12$

$P_6 \Rightarrow$ " " " " " 6^{th} week $= 1 - 0.97$

$P_6 = 0.03$

$$\text{Probability} = P_1 + P_2 + P_3 + P_4 + P_5 + P_6$$
$$= 0.06 + 0.16 + 0.24 + 0.36 + 0.12 + 0.03$$
$$= 1$$

Let N_i be the number of bulbs in the beginning

$N_0 = 1000$

$$N_1 = N_0 P_1$$
$$= 1000 \times 0.09 = 90$$

$$N_2 = N_0 P_2 + N_1 P_1$$
$$= 1000 \times 0.16 + 90 \times 0.9$$
$$= 168$$

$$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$$
$$= 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09$$
$$= 240 + 14.4 + 15.12$$
$$= 269.52$$

$$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$$
$$= 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 269.5 \times 0.09$$
$$= 360 + 21.6 + 26.88 + 24.21$$
$$= 432.69$$

$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$

$= 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 269 \times 0.16 + 432 \times 0.09$

$= 120 + 32.4 + 40.32 + 43.04 + 38.88$

$= 274.64$

$N_6 = N_0P_6 + N_1P_5 + N_2P_4 + N_3P_3 + N_4P_2 + N_5P_1$

$= 1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 269 \times 0.24 + 432 \times 0.16 + 274.64 \times 0.09$

$= 30 + 10.8 + 60.48 + 64.56 + 69.12 + 24.71$

$= 259.67$

Average life of the bulbs $= \Sigma x_i P_i$

$= x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6$

$= 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 + 6P_6$

$= 0.09 + 2 \times 0.16 + 3 \times 0.24 + 4 \times 0.36 + 5 \times 0.12 + 6 \times 0.03$

$= 0.09 + 0.32 + 0.72 + 1.44 + 0.6 + 0.18$

$= 3.35$

Average number of replacements per week $= \dfrac{N_0}{\text{Mean age}}$

$$= \frac{1000}{3.35} = 298.50 = 299$$

End of the week	Total cost of group replacement	Average cost per week
1	$1000 \times 0.70 +$	
2		
3		
4		
5		
6		

UNIT – 4

Inventory Theory

* Inventory :- Inventory is a stock of items which is to be used in future. So in another words the inventory is also called stock.
An inventory consist of a usable, but idle resources such as money, machines and men.

• Inventory Costs – The three costs considered inventory control models are –
1- Inventory carrying cost or stock holding costs.
2- Storage costs.
3- Procurement costs or set-up costs.
4- Total inventory cost.

* Inventory carrying costs or Stock holding costs :- Some components of the stock holding cost are –

1- Cost of money or capital tied up in inventories – More borrowed from the banks may cost interest of about 18%. It is generally taken somewhere around 15% to 20% of the value of the inventories.

2- Cost of Storage space – This consist of rent for space. Besides space expenses, this will also include heating, lighting and other atmospheric control expenses. Typical values may vary from 1 to 3%.

3- Handling costs – Expenditure on stock holding is called handling costs. Such as cost of labour, overhead cranes, gantries and other machinery used for this purpose.

✓ ★ Ordering or Set-up cost :- When an item is produced internally, ordering cost is referred as set-up cost which includes both paper work costs and physical preparation costs.

Ordering cost = (Cost per order or per set-up) X (Number of orders or setups places in the planning period).

✓ ★ Shortage or stock-out and customer-service cost :- The shortage of items occurs when items cannot be supplied on demand. Therefore, shortage costs are usually interpreted in two ways-

1- The supply of items is awaited by the customers i.e., the items are back ordered.

2- Customers are not ready to wait.

This situation may lead to less of customer goodwill and therefore causes loss of sale.

Therefore,

Shortage cost = (cost of being short one unit of an item) X (average number of units cost)

The average number of units short in a planning period is obtain by average number of units short

$$= \left(\frac{\text{minimum shortage} + \text{maximum shortage}}{2}\right) \times \text{period of shortage}$$

Page No :

★ Inventory Control Problem :- The inventory control problem consist of determination of two basic factors-

1- When to order - This is related to the lead time of an item.
There should be sufficient stock for each item so that costomers, order can be reasonably met from this stock until replenishment. This stock level known as reorder level. It is obtain by compromising the cost of maintaining these stocks and the dis-service to the customer if his orders are not filled in time.

2- How much to order - We know each order is related with its ordering cost. To maintain it low, the number of orders should be as few as possible. But large order size would imply high inventory carrying cost. The over problem is determine how must order is solved by compromising.

★ Concept of Economic Ordering Quantity (E.O.Q.) :-
The economic ordering quantity is that size of order which minimizes total annual cost of carrying inventory and cost of ordering under the assumed conditions of certainly and that annual demands are known.

Ques 1- Define the following :-
(i) Transportation problem
(ii) Mathematical formulation of Assignment problem.
(iii) Unbalanced assignment problem.

Sol :- (i) Transportation Problem :-
Transportation problem is a special kind of linear programming problem in which goods are transported from a set of sources to a set of destination subject to the supply and demand of the source and destination, respectively, such that the total cost of transportation is minimized.

Let m = the number of sources.
n = the number of destinations.
a_i = the supply at the source i.
b_j = the demand at the destination j.
C_{ij} = the cost of transportation per unit from ith source to jth destination.
x_{ij} = the number of units to be transported from the source ith to the jth destination.

(ii) Mathematical formulation of Assignment Problem :-
Mathematically, the assignment problem can be stated as -
Minimize the total cost:

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

where $i = 1, 2, 3 \ldots\ldots n$
$j = 1, 2, 3 \ldots\ldots n$

Subject to restrictions

$$x_{ij} = \begin{cases} 1, & \text{if ith person is assigned jth job} \\ 0, & \text{if not.} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (one job is done by the ith person } i = 1,2,3 \ldots n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ (only one person should be assigned the jth job, } j = 1,2,3 \ldots n)$$

where x_{ij} denotes the jth job is to be assigned to the ith person.

(iii) **Unbalanced Assignment Problem :-**

If the cost matrix of an assignment problem is not square matrix, the assignment problem is called unbalanced assignment problem. In such cases add dummy rows or columns with zero costs. Then the usual assignment algorithm can be applied to this resulting balanced problem.

Example –

Job

Person	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24

Job

Person	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D (dummy)	0	0	0	0

Ques 2:- Solve the following transportation problem by using -

(i) Matrix minima method

(ii) VAM

(iii) Test for optimality.

Destination

		D_1	D_2	D_3	D_4	Supply
	I	1	2	1	4	30
Origin	II	3	3	2	1	50
	III	4	2	5	9	20
Demand		20	40	30	10	100

Sol:- (i) By matrix minima method-

Destination

		D_1	D_2	D_3	D_4	Supply
	I	[20] 1	2	[10] 1	4	30/10
Origin	II	3	[20] 3	[20] 2	[10] 1	50/40/20/0
	III	4	[20] 2	5	9	20/0
Demand		20/0	40/20/0	30/20/0	10/0	100

Minimum cost $= 1\times20 + 1\times10 + 3\times20 + 2\times20 + 1\times10 + 2\times20$

$= 20 + 10 + 60 + 40 + 10 + 40$

$= 180$ Rs.

Initial Basic Feasable Solution -

$x_{11} = 20$

$x_{13} = 10$

$x_{22} = 20$

$x_{23} = 20$

$x_{24} = 10$

$x_{32} = 20$

(ii) By VAM -

	D_1	D_2	D_3	D_4	Supply	Penalty
I	1	2	1	4	30	1
II	3	3	2	$\boxed{10}$ 1	50/40	1
III	4	2	5	9	20	1
Demand	20	40	30	10/0	100	
Penalty	2	1	1	3 (max)		

	D_1	D_2	D_3	Supply	Penalty
I	$\boxed{20}$ 1	2	1	30/10	1
II	3	3	2	40	1
III	4	2	5	20	1
Demand	20/0	40	30	90	
Penalty	2 (max)	1	1		

	D_2	D_3	Supply	Penalty
I	2	$\boxed{10}$ 1	10/0	1 ← (max)
II	3	2	40	1
III	2	5	20	1
Demand	40	30/20	70	
Penalty	1	1		

	D_2	D_3	Supply	Penalty
II	$\boxed{20}$ 3	$\boxed{20}$ 2	40/20	1
III	$\boxed{20}$ 2	5	20/0	3 ← (max)
Demand	40/20	20/0	60	
Penalty	1	3		

Minimum cost $= 1 \times 10 + 1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 2 \times 20$

$= 10 + 20 + 10 + 60 + 40 + 40$

$= 180$ Rs.

Initial Basic Feasable Solution —

$x_{11} = 20$	$x_{23} = 20$
$x_{13} = 10$	$x_{24} = 10$
$x_{22} = 20$	$x_{32} = 20$

(iii) To test the optimality —

	D_1	D_2	D_3	D_4		
I	$\boxed{20}$ 1	2	$\boxed{10}$ 1	4	30	u_1
II	3	$\boxed{20}$ 3	$\boxed{20}$ 2	$\boxed{10}$ 1	50	u_2
III	4	$\boxed{20}$ 2	5	9	20	u_3
	20	40	30	10	100	
	v_1	v_2	v_3	v_4		

$m + n - 1 = 3 + 4 - 1 = 6 =$ Occupied cells

Calculate for occupied cells —

$C_{ij} = u_i + v_j$

$C_{11} = u_1 + v_1 \Rightarrow 1 = u_1 + v_1$

$C_{13} = u_1 + v_3 \Rightarrow 1 = u_1 + v_3$

$C_{22} = u_2 + v_2 \Rightarrow 3 = u_2 + v_2$

$C_{23} = u_2 + v_3 \Rightarrow 2 = u_2 + v_3$

$C_{24} = u_2 + v_4 \Rightarrow 1 = u_2 + v_4$

$C_{32} = u_3 + v_2 \Rightarrow 2 = u_3 + v_2$

Put $u_2 = 0$

$v_2 = 3$	$u_1 = -1$
$v_3 = 2$	$u_3 = -1$
$v_4 = 1$	$v_1 = 2$

Calculate for non-occupied cells —

$d_{ij} = C_{ij} - (u_i + v_j)$

$d_{12} = 2 - (u_1 + v_2) \Rightarrow 2 - (-1 + 3) = 0$

$d_{14} = 4 - (u_1 + v_4) \Rightarrow 4 - (-1 + 1) = 4$

$d_{21} = 3 - (u_2 + v_1) \Rightarrow 3 - (0 + 2) = 1$

$$d_{31} = 4-(u_3+v_1) \Rightarrow 4-(-1+2) = 3$$
$$d_{33} = 5-(u_3+v_3) \Rightarrow 5-(-1+2) = 4$$
$$d_{34} = 9-(u_3+v_4) \Rightarrow 9-(-1+1) = 9$$

$$d_{ij} \geq 0$$

The basic feasable solution is optimal.
Minimum transportation cost = Rs. 180.

Ques 3:- A company has 4 machines to do 3 jobs. Each can be assigned to one and only one machine. The cost of each job on each machine is given in the following table –

Machine

		W	X	Y	Z
Job	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

Sol:- Since the given problem is an unbalanced assignment problem, so we add a dummy row.

	W	X	Y	Z
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22
D	0	0	0	0

Now the problem can be solved by usual method.

Step 1- Subtract the smallest element of each row from every element of the corresponding row in the matrix

	W	X	Y	Z
A	0	6	10	14
B	0	5	9	11
C	0	5	9	12
D	0	0	0	0

Step 2 – Make the assignment.

	W	X	Y	Z	
A	$\boxed{0}$	6	10	14	✓
B	$\cancel{0}$	5	9	11	✓
C	$\cancel{0}$	5	9	12	✓
D	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
	✓				

Step 3 –

	W	X	Y	Z	
A	$\boxed{0}$	1	5	9	✓
B	$\cancel{0}$	$\boxed{0}$	4	6	✓
C	$\cancel{0}$	$\cancel{0}$	4	7	✓
D	5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	
	✓	✓			

Step 4 –

	W	X	Y	Z
A	$\boxed{0}$	1	1	5
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
D	9	4	$\cancel{0}$	$\boxed{0}$

$A \rightarrow W, \; B \rightarrow X, \; C \rightarrow Y, \; D \rightarrow Z$

Total minimum cost = ~~18 + 13 + 19 + 0~~

~~= 50 Rs.~~

Ques 4: Find the optimal solution for the assignment problem –

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

Sol :- Step 1 – Subtract the smallest element of each row from every element of the corresponding row in the given matrix.

$$
\begin{array}{c|ccccc|}
 & I & II & III & IV & V \\
\hline
A & 3 & 9 & 0 & 8 & 12 \\
B & 3 & 1 & 6 & 0 & 9 \\
C & 1 & 4 & 3 & 0 & 4 \\
D & 4 & 7 & 0 & 11 & 9 \\
E & 4 & 0 & 2 & 1 & 5
\end{array}
$$

Step 2 – Subtract the smallest element of each column from every element of the corresponding column in the given matrix.

$$
\begin{array}{c|ccccc|}
 & I & II & III & IV & V \\
\hline
A & 2 & 9 & 0 & 8 & 8 \\
B & 2 & 1 & 6 & 0 & 5 \\
C & 0 & 4 & 3 & 0 & 0 \\
D & 3 & 7 & 0 & 11 & 5 \\
E & 3 & 0 & 2 & 1 & 1
\end{array}
$$

Step 3 – Make the assignment.

$$
\begin{array}{c|ccccc|c}
 & I & II & III & IV & V & \\
\hline
A & 2 & 9 & \boxed{0} & 8 & 8 & \checkmark \\
B & 2 & 1 & 6 & \boxed{0} & 5 & \\
C & \boxed{0} & 4 & 3 & \cancel{0} & \cancel{0} & \\
D & 3 & 7 & \cancel{0} & 11 & 5 & \checkmark \\
E & 3 & \boxed{0} & 2 & 1 & 1 & \\
\hline
 & & & \checkmark & & &
\end{array}
$$

Step 4 -

	I	II	III	IV	V
A	$\boxed{0}$	7	~~0~~	6	6
B	2	1	8	$\boxed{0}$	5
C	~~0~~	4	5	~~0~~	$\boxed{0}$
D	1	5	$\boxed{0}$	9	3
E	3	$\boxed{0}$	4	1	1

$A \to I$, $B \to IV$, $C \to V$, $D \to III$, $E \to II$

Total minimum cost $= 11 + 6 + 16 + 17 + 10$

$= 60$ Rs.

Ans.

3/4/19

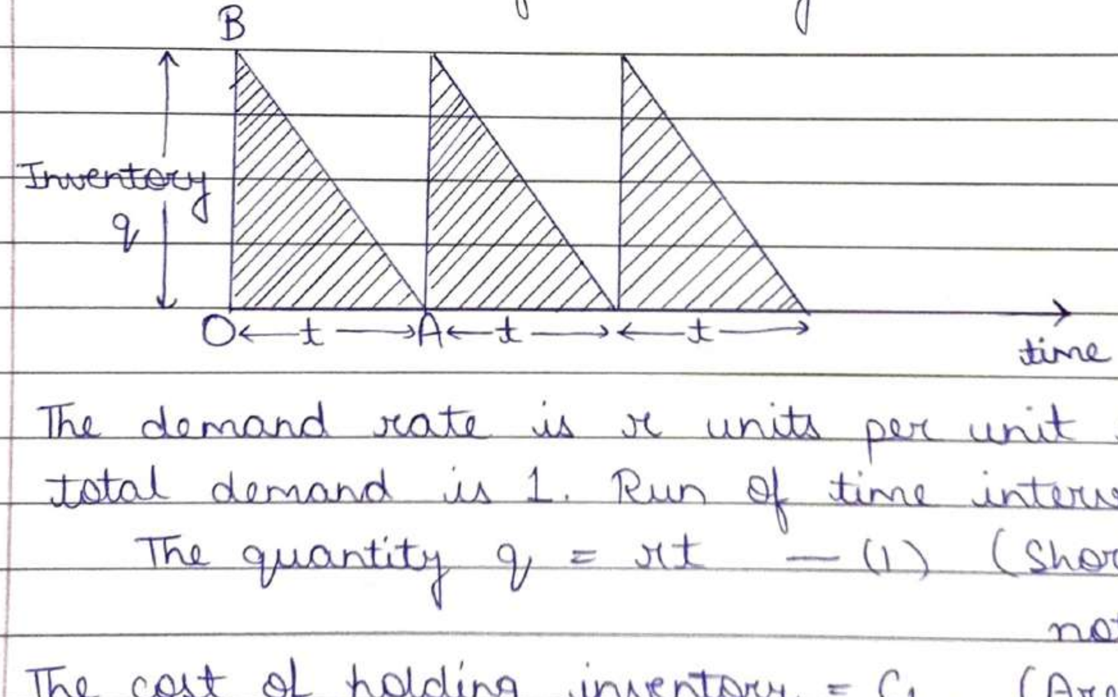
* Model I :-

To drive economic lot size formula and the minimum average cost under the following assumptions -

1- Demand is uniform at a rate.
2- Production is instantaneous (i.e. production rate ~~of~~ is infinite.
3- Lead time is 0.
4- C_1 = holding cost per unit of quantity per unit of time.
5- C_3 = set-up cost per production run.
6- Shortage are not allow.

Proof - Let q be the units of quantity produced per production run at interval of time t.

The situation of inventory -



The demand rate is r units per unit time. The total demand is 1. Run of time interval t is rt.

The quantity $q = rt$ — (1) (Shortage are not allowed)

The cost of holding inventory $= C_1$ (Area of OAB)

$$= C_1 \times \frac{1}{2} q t$$

Set-up cost $= C_3$

The total cost per production run of time t

$$= \frac{1}{2} C_1 q t + C_3$$

The average total cost per unit time $C(q)$

$$= \frac{1}{2} C_1 q + \frac{C_3}{t}$$

From (1), $t = \frac{q}{r}$

$$C(q) = \frac{1}{2} C_1 q + \frac{C_3 r}{q} \quad — (2)$$

This equation is known as cost equation.
For minimum value-

$$\frac{dC}{dq} = \frac{1}{2} C_1 - \frac{C_3 r}{q^2}$$

Put $\frac{dC}{dq} = 0$

$$\frac{1}{2} C_1 - \frac{C_3 r}{q^2} = 0$$

$$\frac{1}{2} C_1 = \frac{C_3 r}{q^2}$$

$$q^2 = \frac{2 C_3 r}{C_1}$$

$$\boxed{q = \sqrt{\frac{2 C_3 r}{C_1}}}$$

Again diff. $\frac{d^2 C}{dq^2} = \frac{2 C_3 r}{q^3}$ (+ve)

For minimum $\boxed{q = q^* = \sqrt{\frac{2 C_3 r}{C_1}}}$

This is the economic lot size formula.
Put in eq (1)

$$t = \frac{q}{r}$$

$$t = \frac{1}{r} \sqrt{\frac{2 C_3 r}{C_1}}$$

$$t = \frac{1}{r} \times \sqrt{r} \sqrt{\frac{2 C_3}{C_1}}$$

$$t = \sqrt{\frac{2C_3}{C_1 r}}$$

$$\boxed{t = t^* = \sqrt{\frac{2C_3}{C_1 r}}}$$

The minimum cost per unit time is given by-
From eq(2)

$$C_{min} = \frac{1}{2} C_1 q + \frac{C_3 r}{q}$$

$$C_{min} = \frac{1}{2} C_1 \sqrt{\frac{2C_3 r}{C_1}} + C_3 r \sqrt{\frac{C_1}{2C_3 r}}$$

$$C_{min} = \frac{1}{2} C_1 \frac{\sqrt{2C_3 r}}{\sqrt{C_1}} + C_3 r \frac{\sqrt{C_1}}{\sqrt{2} \times \sqrt{C_3 r}}$$

$$C_{min} = \frac{1}{2} \sqrt{C_1} \sqrt{2C_3 r} + \sqrt{C_3 r} \times \frac{\sqrt{C_1}}{\sqrt{2}}$$

$$C_{min} = \frac{1}{\sqrt{2}} \sqrt{C_1 C_3 r} + \frac{1}{\sqrt{2}} \sqrt{C_1 C_3 r}$$

$$C_{min} = \frac{2}{\sqrt{2}} \sqrt{C_1 C_3 r}$$

$$\boxed{C_{min} = \sqrt{2 C_1 C_3 r}}$$

If C_1 and C_3 are constant then the minimum cost per unit time is propotional to the square root of the demand rate.

Ques - A manufacturer has to supply his customer with 600 units of his product per year. Shortages are not allowed and the shortage cost amount to Rs. 0.60 per unit per year. The set-up cost per run is Rs. 80.00. Find the optimum run size and the minimum average yearly cost.

Sol:- $C_1 = 0.60$ Rs.
$C_3 = 80.00$ Rs.

$r = 600$

$$q = \sqrt{\frac{2C_3 r}{C_1}} = \sqrt{\frac{2 \times 80 \times 6\cancel{00} \times 100}{0.6\cancel{0}}} = \sqrt{160000}$$

$$\boxed{q = 400}$$

$$t = \frac{q}{r} = \frac{400}{600} = \frac{2}{\cancel{3}} \times \cancel{12}^{4}$$

$t = 8$ months

$$C_{min} = \sqrt{2C_1C_3 r}$$

$$= \sqrt{\frac{2 \times 0.60}{100} \times 80 \times 600} = \sqrt{57600}$$

$C_{min} = 240$ unit / year

Ques - The shortage cost of one item is Rs. 1.00 per month and the setup cost is Rs. 25.00 per run. If the production is instantaneous and the demand is 200 unit per month. Find the optimal size of the batch and best time for the replacement of inventory.

Sol:

$C_1 =$ Rs. 1

$C_3 =$ Rs. 25

$r = 200$

$$q = \sqrt{\frac{2C_3 r}{C_1}} = \sqrt{\frac{2 \times 25 \times 200}{1}} = \sqrt{10000}$$

$$\boxed{q = 100}$$

$$t = \frac{q}{r} = \frac{100}{200} = \frac{1}{2} \text{ month}$$

$$C_{min} = \sqrt{2C_1C_3 r}$$

$$= \sqrt{2 \times 1 \times 25 \times 200} = 100 \text{ unit / month}$$

$$C_{min} = 100 \times 12$$

$$= 1200 \text{ unit / year.}$$

★ Another form of model I :- To drive an economic lot size formula for the following assumptions -

(i) λ = the demand for product in one unit of time (say 1 year).

(ii) Production rate is infinite.

(iii) Lead time is zero.

(iv) P = Price of one time of product in rupees.

(v) I = Cost of carrying one rupee to the inventory for one year.

(vi) C_3 = Set-up cost per order (per cycle).

(vii) Shortages not allowed.

Sol:- Let q be the units of quantity production per cycle (in time t).

From model I, total inventory in one cycle

$$= \frac{1}{2} q t.$$

The average inventory $\frac{q}{2}$. It will continue through out the year holding cost per item for 1 year

$$= IP$$

The total holding cost per year $= \frac{1}{2} q IP$

Setup cost in one year $= \frac{\lambda}{q} C_3$

The total cost in one year $C(q) = \frac{1}{2} q IP + \frac{\lambda}{q} C_3$

For minimum $\frac{dC}{dq} = \frac{1}{2} IP + \frac{\lambda}{q^2} C_3$

Now put $\frac{dC}{dq} = 0$

$$\frac{1}{2} IP - \frac{\lambda}{q^2} C_3 = 0$$

$$\frac{\lambda}{q^2} C_3 = \frac{1}{2} IP$$

$$q = \sqrt{\frac{2\lambda C_3}{IP}}$$

again diff. $\frac{d^2c}{dq^2} = \frac{2\lambda}{q^3} C_3$ which is positive

★ Model II :-

$$q = \sqrt{\frac{2RC_3}{C_1 t}}$$

R is total demand in the total period.
It is the economic lot size formula.
Minimum cost per unit time :

$$C_{min} = \frac{1}{2} C_1 \sqrt{\frac{2RC_3}{C_1 t}} + \frac{RC_3}{t} \sqrt{\frac{C_1 t}{2RC_3}}$$

$$C_{min} = \sqrt{\frac{2 C_1 C_3 R}{t}}$$

★ Model III :-
Economic lot size formula

$$q = q^* = \sqrt{\frac{2 C_3 r k}{C_1 (k-r)}}$$

Minimum cost per unit time is given by

$$C_{min} = \sqrt{2 C_1 C_3 r \left(\frac{k-r}{r}\right)}$$

Also time of one run is given by $t = \sqrt{\frac{2 C_3 k}{C_1 r (k-r)}}$

Ques- A manufacturer has to supply 12000 units of a product per year to each customer. The demand is fixed and known. Shortage cost is assumed to be infinite. The inventory holding cost is Rs 0.20 per unit / month. and the setup cost per run is Rs. 350, then find-

(i) The optimum run size q_0

(ii) Optimum scheduling period t_0

(iii) Minimum total variable yearly cost.

Sol:- $C_1 = 0.20$ Rs. $\qquad r = \frac{\cancel{12000}^{1000}}{\cancel{12}} = 1000$ / month

$C_3 = 350$ Rs.

(i) $q_0 = \sqrt{\frac{2C_3 r}{C_1}} = \sqrt{\frac{2 \times 350 \times 1000 \times 10\cancel{0}}{0.2\cancel{0}}}$

$= \sqrt{3500000} = 1870.8$ units / run

(ii) $t_0 = \frac{q}{r}$

$t_0 = \frac{1870.8}{1000} = 1.870$

(iii) $C_{min} = \sqrt{2 C_1 C_3 r}$

$= \sqrt{2 \times \frac{0.2\cancel{0}}{10\cancel{0}} \times 35\cancel{0} \times (1000 \cancel{\times 12})}$

$= \sqrt{140000}$

$= 374.16 \times 12$

C_{min} = Rs. 4490 / year

UNIT – 5
Job Sequencing

* Job Sequencing :– The selection of an appropriate order in which to service waiting customers is called sequencing.

A general sequencing problem may be defined as follows–

Let there are n jobs (1, 2, 3 ... n) which have to be processed one at a time at each of m machine A, B, C ..... The order of the machine is given for each job in which it should go to the machine A, B, C ....., The order of the machine is given. The time required by the jobs on each of the machine is also given. Then the problem is to find the sequence out of $(n!)^m$ sequences, which optimizes (minimize) the total time elapsed from the start of the first job to the completion of the last job.

Mathematically,

Let A_i = time required for job i on machine A.
B_i = time required for job i on machine B.
T = total elapsed time for jobs 1, 2, 3 ... n i.e. the time from start of the first job to completion of the last job.

The problem is to determine a sequence $(i_1, i_2 \dots i_n)$ where $(i_1, i_2 \dots i_n)$ is the permutation of integers which will minimize T.

Analytic method have been developed for solving only four simple cases.

1. n jobs and 2 machines A and B, All jobs

processed in the order AB.

2- n jobs and 3 machines A, B and C. All jobs processed in the order ABC.

3- n jobs and m machines A, B, C.... M. All jobs processed in the order ABC.... M.

4- 2 jobs and m machines. Each job to be processed through the machine in a prescribed order.

* Total elapsed time :- This is the time between starting the first job and completing the last job. It is denoted by T.

* Idle time on a machine :- This is the time for which a machine remains idle during the total elapsed time.

* Processing time :- It means the time required by each job on each machine.

* Processing order :- The order in which various machines are required for completing the job.

* <u>Sequencing Decision Problem for n-job on two machines (Johnson's Method)</u> :-

Here we consider the problem of processing n-jobs (1, 2, 3 .... n) on two machines A and B under the following assumptions -

1- Each job is processed in the order AB.

2- A_i = Processing time of i^{th} job on machine A. $(i = 1, 2, 3 .... n)$

3- B_i = Processing time of i^{th} job on machine B. $(i = 1, 2, 3 .... n)$

The problem is to find the sequence of jobs to be performed on two completion of the last job is minimize.
The procedure for the solution of the above problem was developed by Johnson and Bellman. The method is based on minimizing the idle time for second machine. The Johnson's procedure for determine an optimal sequence is a follows-

Step 1- Examine the A_i and B_i for $i=1,2,3 \ldots n$ and select the minimum of these. If there are two or more minimum processing time then select any one of them arbitrarily.

Step 2- If the minimum processing time is for machine A, process that job and place it at the beginning of the sequence.
If the minimum processing time is for machine B, process job first and place it at the end of the sequence.

Step 3- Cross all the jobs already assign and repeat step 1 and 2, placing the remaining jobs next to first or next to last, until all the jobs have been assign.

Step 4- Calculate the time at which each job in the sequence will be processed on machine A. This time can be calculated as follows-
Time at which i^{th} job in a sequence finish on machine A = time when the $(i-1)^{th}$ job in a sequence finish on machine A and the time for start of first job on machine A is zero.

Step 5- Calculate the time at which each job in the sequence will start and finish on machine B as follows –

(i) Time when first job in a sequence start on machine B = time when the first job in a sequence finish on machine A.

(ii) Time the i^{th} job in the sequence finish at B = time when the i^{th} job in a sequence start on machine B + the processing time of i^{th} job on machine B. ($i = 1, 2, 3 \ldots n$).

(iii) Time at which the $(i+1)^{th}$ job in a sequence finish on machine B = maximum time when the $(i+1)^{th}$ job in a sequence finish on the machine at the time when the i^{th} job in a sequence finish on machine B. ($i = 1, 2, 3 \ldots n$).

Step 6 – Calculate the total elapsed time to process all jobs through two machines i.e., when the n^{th} job in a sequence finish on machine B.

Step 7 – Compute the idle time for machine A and B.

Idle time for machine A = time for n^{th} job in a sequence finish on machine B.

Idle time for machine B = time at which the first job in a sequence finish on machine A + Time i^{th} job in a sequence start on machine B – Time when $(i-1)^{th}$ job in a sequence finish on machine B.

Ques- A company has 3 jobs on hand. Each of these must be processed from two departments. The sequential order for which is-

Department A - Press shop

Department B - Finishing

The table below list the number of days required by each job in each department -

	Job I	Job II	Job III
Department A →	8	6	5
Department B →	8	3	4

Find the sequence in which the 3 jobs should be processed so as to take minimum time to finish all the 3 jobs.

Sol:- Smallest time in the given process-

I	III	II

Jobs	Department A		Department B		Ideal time of B
	Time in	Time out	Time in	Time out	
I	0	8	8	16	8
III	8	13	16	20	
II	13	$\boxed{19}$	20	$\boxed{23}$	

Total minimum elapsed time = 23 days

Ideal time of department A = 23 - 19

= 4 days.

* Travelling salesman problem :- Travelling salesman problem is very similar to the assignment problem except that in the formal there is an additional restrictions that a salesman who start

from his home city, visit each city only once and return to his home city. The problem is to find to routes shortest in distance (time or cost). If the number of cities is 3 (A, B and C) of which starting base A there are two possible routes $A \to B \to C$ and $A \to C \to B$. In general for n cities there are $(n-1)!$ possible routes.

Mathematically the problem may be stated as follows-

$$\text{Minimize} \sum_i \sum_j c_{ij} x_{ij} \quad *$$

Ques- There are 7 jobs, each of which has to go through the machine A and B in the order AB. Processing time hours are given as-

Job	1	2	3	4	5	6	7
Machine A	3	12	15	6	10	11	9
Machine B	8	10	10	6	12	1	3

Determine a sequence of these jobs that will minimize the total elapsed time T.

Sol:- We obtain the sequence

1	5	2	3	4	7	6

	Machine A		Machine B		Idal time of
Job	Time in	Time out	Time in	Time out	machine B
1	0	3	3	11	3
4	3	9	11	17	0
5	9	19	19	31	2
3	19	34	34	44	3
2	34	46	46	56	2
7	46	55	56	59	0
6	55	$\boxed{66}$	66	$\boxed{67}$	7

Total elapsed time = 67
Ideal time = 67-66 = 1 hour

Ques –

Job	1	2	3	4	5	6
Machine A	30	120	50	20	90	110
Machine B	80	100	90	60	30	10

Sol:- Sequence

4	1	3	2	5	6

	Machine A		Machine B		
Job	Time in	Time out	Time in	Time out	Ideal time
4	0	20	20	80	20 (B)
1	20	50	80	160	0
3	50	100	160	250	0
2	100	220	250	350	0
5	220	310	350	380	0
6	310	$\boxed{420}$	380	~~390~~ $\boxed{430}$	40

Total elapsed time = 430 hrs
Ideal time for A = 430 – 420 = 10 hrs
" " " B = 20 + 40 = 60

*• Subject to the additional constraint that x_{ij} is to be so chosen that no city is visited twice before the tour of all the cities is completed.
The travelling salesman problem can be written in the form of square assignment problem.

Form

	1	2	3	----	n
1	∞	C_{12}	C_{13}	----	C_{1n}
2	C_{21}	∞	C_{23}	----	C_{2n}
3	C_{31}	C_{32}	∞	----	C_{3n}
⋮	⋮			∞	
n	C_{n1}	C_{n2}	----	----	∞

Ques- Solve the following salesman problem –

∞	4	10	14	2
12	∞	6	10	4
16	14	∞	8	14
24	8	12	∞	10
2	6	4	16	∞

∞	2	8	12	0
8	∞	2	6	0
8	6	∞	0	6
16	0	4	∞	2
0	4	12	14	∞

	A	B	C	D	E
A	∞	2	16	12	$\boxed{0}$
B	8	∞	$\boxed{0}$	6	$\cancel{0}$
C	8	6	∞	$\boxed{0}$	6
D	16	$\boxed{0}$	2	∞	2
E	$\boxed{0}$	4	$\cancel{0}$	14	∞

It is a feasable solution and re-check sales man conditions.

$A \rightarrow E \rightarrow A$ condition is not satisfied.

General rule:

1- If there is no option then select the next lowest cost in the matrix.

2- First reference must be to 0.

3- After 0 the next preference is to 1 or 2.
4- If there is no option then it may be high value.

	A	B	C	D	E
A	∞	$\boxed{2}$	6	12	$\cancel{8}$
B	8	∞	$\boxed{0}$	6	$\cancel{8}$
C	8	6	∞	$\boxed{0}$	6
D	16	$\cancel{8}$	$\cancel{8}$	∞	$\boxed{2}$
E	$\boxed{0}$	4	$\cancel{8}$	14	∞

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$

★ Money value :- Money value is the value of money to be spent after the period, on the original value with some interest.
The value of money change with time. Since has value over time. This can be explain with the following example -

Ex- If we borrow ₹100 at the rate of interest 10% per year and spend this amount today, then we have to pay ₹110 after one year.

★ Present value / Present Worth Factor :-
Present value is the current value of money or the running cost of an item that deteriorates over a period of time.
If r is the rate of interest then $\frac{1}{(1+r)^n}$ is called the present value factor of one rupee spent in n years.

★ Discount rate (Depreciation value) :-

Discount rate refers to the interest rate used to determine the present value of future cash flows. It is not just the time value of money but the risk or uncertainity of future cash flows.

$$v = \frac{1}{(1+r)}$$