DMS: DISCRETE MATHEMATICAL STRUCTURES

SET THEORY

Set Theory is a branch of Mathematics that studies the collection of objects and operations based on it. Sets are studied under the mathematical logic division of mathematics. A set is simply a collection of objects. The words collection, aggregate, and class are synonymous. On the other hand element, members, and objects are synonymous and stand for the members of the set of which the set is comprised..

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Set definition

Sets are defined as"a well-defined collection of objects". Let's say we have a set of Natural numbers then it will have all the natural numbers as its member and the collection of the numbers is well defined that they are natural numbers. A set is always denoted by a capital letter. For Example, a set of Natural Numbers is given by N = {1, 2, 3, 4…..}. The term 'well defined' should be taken care of, as if we try to make a set of best players, then the term 'best' is not well defined. The concept of best, worst, beautiful, powerful, etc. varies according to the notions, assumptions, likes, dislikes, and biases of a person.

Elements of a Set

The objects contained by a set are called the elements of the set. For Example, in the set of Natural Numbers 1, 2, 3, etc. are the objects contained by the set, hence they are called the elements of the set of Natural Numbers. We can also say that 1 belongs to set N. It is represented as $1 \in N$, where $\in$ is the symbol of belongs to.

Cardinal Number of a Set

The number of elements present in a set is called the Cardinal Number of a Set. For Example, let's say P is a set of the first five prime numbers given by P = {2, 3, 5, 7, 11}, then the Cardinal Number of set P is 5. The Cardinal Number of Set P is represented by n (P) or |P| = 5.

Example of Sets

A Set is a well-defined collection of objects. The objects inside a set are called members of the set. Some examples of sets are mentioned below:

Set of Natural Numbers: N = {1, 2, 3, 4….}
Set of Even Numbers: E = {2, 4, 6, 8…}
Set of Prime Numbers: P = {2, 3, 5, 7,….}
Set of Integers: I = {…, -4, -3, -2, -1, 0, 1, 2,….}

Some Standard Sets used in Set Theory

- Set of Natural Numbers is denoted by **N**
- Set of Whole Numbers is denoted by **W**
- Set of Integers is denoted by **Z**
- Set of Rational Numbers is denoted by **Q**
- Set of Irrational Numbers is denoted by **T**
- Set of Real Numbers is denoted by **R**

Representation of Sets

Sets are primarily represented in two forms

- Roster Form
- Set Builder Form

Roster Form

In the Roster Form of the set, the elements are placed inside braces {} and are separated by commas. Let's say we have a set of the first five prime numbers then it will be represented by P = {2, 3, 5, 7, 11}. Here the set P is an example of a finite set as the number of elements is finite, however, we can come across a set that has infinite elements then in that case the roster form is represented in the manner that some elements are placed followed by dots to represent infinity inside the braces. Let's say we have to represent a set of Natural Numbers in Roster Form then its Roster Form is given as N = {1, 2, 3, 4…..}.

Set Builder Form

In Set Builder Form, a rule or a statement describing the common characteristics of all the elements is written instead of writing the elements directly inside the braces. For Example, a set of all the prime numbers less than or equal to 10 is given as

P = {p : p is a prime number ≤ 10}. In another example, the set of Natural Numbers in set builder form is given as N = {n : n is a natural number}.

Representation of Sets

Types of Sets

There are different types of sets categorized on various parameters. The different types of sets are explained below:

Empty Set

A set that has no elements inside it is called an Empty Set. It is represented by Φ or {}. For Example A = {x : x ∈ N and 2 < x < 3}. Here, between 2 and 3, no natural number exists, hence A is an Empty Set. Empty Sets are also known as Null Sets.

Singleton Set

A set that has only one element inside it is called a Singleton Set. For Example, B = {x : x ∈ N and 2 < x < 4} = {3}. Here between 2 and 3 only one element exists, hence B is called a Singleton Set.

Finite Set

A set that has a fixed or finite number of elements inside it is called a Finite Set. For Example A = {x : x is an even number less than 10} then A = {2, 4, 6, 8}. Here A has 4 elements, hence A is a finite set. The number of elements present in a finite set is called the Cardinal Number of a finite set. It is given as n(A) = 4.

Infinite Set

A set that has an indefinite or infinite number of elements inside it is called a Finite Set. For Example A = {x : x is an even number l} then A = {2, 4, 6, 8……}. Here A has unlimited elements, hence A is an infinite set.

Equivalent Sets

If the number of elements present in two sets is equal i.e. the cardinal number of two finite sets is the same then they are called Equivalent Sets. For Example, A = {x : x is an even number up to 10} = {2, 4, 6, 8, 10} and B = {y : y is an odd number less than 10} = {1, 3, 5, 7, 9}. Here, the cardinal number of set A is n(A) = 5 and that of B is given as n(B) = 5 then we see that n(A) = n(B). Hence A and B are equivalent sets.

Equal Sets

If the number of elements and also the elements of two sets are the same irrespective of the order then the two sets are called equal sets. For Example, if set A = {2, 4, 6, 8} and B ={8, 4, 6, 2} then we see that number of elements in both sets A and B is 4 i.e. same and the elements are also the same although the order is different. Hence, A and B are Equal Sets. Equal Sets are represented as A = B.

Unequal Sets

If at least any one element of one set differs from the elements of another set then the two sets are said to be unequal sets. For Example, if set A = {2, 4, 6, 8} and B = {4, 6, 8, 10} then set A and B are unequal sets as 2 is present in set A but not in B and 10 is present in set B but not in A. Hence, one element differs between them thus making them unequal. However, the cardinal number is the same therefore they are equivalent sets.`

Overlapping Sets

If at least any one element of the two sets are the same then the two sets are said to be overlapping sets. For Example, if set A = {1, 2, 3} and set B = {3, 4, 5} then we see that 3 is the common element between set A and set B hence, set A and set B are Overlapping Sets.
Disjoint Sets

If none of the elements between two sets are common then they are called the Disjoint Sets i.e., for two sets A and B if A. For Example, set A = {1, 2, 3} and set B = {4, 5, 6} then we observe that there is no common element between set A and set B hence, set A and B are Disjoint Sets.

Apart from the above-mentioned sets, there are other sets called, Subsets, Supersets, Universal Sets, and Power Sets. We will learn them below in detail.

Subsets

If A and B are two sets such that every element of set A is present in set B then A is called the subset of B. It is represented as A ⊆ B and read as ‘A is a subset of B’. Mathematically it is expressed as

$\boldsymbol{A \subseteq B \text{ iff}}$

$\boldsymbol{a \in A \Rightarrow a \in B}$

If A is not a subset of B we write it as A ⊄ B.

For Example, if A = {1, 2} and B = {1, 2, 3} then we see that all the elements of A are present in B, hence $A \subseteq B$. There are two kinds of subset Proper Subset and Improper Subset.

Proper Subset

If a subset doesn't contain all the elements of the set or has fewer elements than the original set then it is called the proper subset. For example, in set A = {1, 2} and B = {1, 2, 3}, the subset A doesn't contain all the elements of the original set B, hence A is a proper subset of B. It is represented as $A \subset B$. Empty set is a proper subset of a given set as it has no elements.

Improper Subset

If a subset contains all the elements that are present in the original set then it is called an Improper Subset. For Example, if set A = {1, 2, 3, 4} and set B = {1, 2, 3, 4} then A is the improper subset of set B. It is mathematically expressed as $A \subseteq B$. Thus we deduce that two sets are equal iff $A \subseteq B$ and $B \subseteq A$. It should be noted that an empty set is an improper subset of itself.

Some Important Results on Subset

- Every set is a subset of itself
- An empty Set is a subset of every set.
- The number of possible subsets for a given finite set with 'n' number of elements is equal to 2n.
- $N \subset W \subset Z \subset Q \subset R$ and $T \subset R$ where N is a set of Natural Numbers, W is a set of Whole Numbers, Z is a set of integers, Q is a set of Rational Numbers, T is a set of irrational numbers and R is set of real numbers.

Superset

If all the elements of set A are present in set B then set B is called the Superset of set A. It is represented as $B \supseteq A$. Let's say if A = {2, 3, 4} and B = {1, 2, 3, 4} then we see that all elements of set A are present in set B, hence $B \supseteq A$. If a superset has more elements than its subset then it is called a proper or strict superset. A Proper Superset is represented as $B \supset A$. Some of the Properties of Supersets are mentioned below:

- Every set is a superset of itself.
- Every set is a superset of an empty set.

- Total number of possible supersets for a given subset is infinite
- If B is a superset of A then A is a subset of B

Universal Set

The set that contains all the sets in it is called a Universal Set. Let's say set A = {1, 2, 3}, set B = {4, 5}, and set C = {6, 7} then Universal Set is given as U = {1, 2, 3, 4, 5, 6, 7}. Another Example of a Universal Set is U = {Set of All Living Beings} then which includes both floras and faunas. Flora and fauna are the subsets of Universal Sets U.

Power Set

A set that contains all the subsets as its element is called the Power Set. For Example, if set A = {1, 3, 5} then its subsets are {Φ}, {1}, {2}, {3}, {1, 3}, {3, 5}, {1, 5} and {1, 3, 5} then its Power Set is given as P(A) = {{Φ}, {1}, {2}, {3}, {1, 3}, {3, 5}, {1, 5}, {1, 3, 5}}. As we know the number of possible subsets for a finite set containing n elements is given by 2n then the number of elements in the power set is also 2n

Set Theory Symbols

There are various symbols that are used in Set Theory. The symbol and their explanation are tabulated below:

Symbol	Explanation
{}	Set
$x \in A$	x is an element of set A
$x \notin A$	x is not an element of set A
$\exists$ or $\nexists$	There exist or there doesn't exist
Φ	Empty Set

Symbol	Explanation
A = B	Equal Sets
n(A)	Cardinal Number of Set A
P(A)	Power Set
$A \subseteq B$	A is a subset of B
$A \subset B$	A is the Proper subset of B
$A \nsubseteq B$	A is not a subset of B
$B \supseteq A$	B is the superset of A
$B \supset A$	B is a proper superset of A
$B \nsupseteq A$	B is not a superset of A
$A \cup B$	A union B
$A \cap B$	A intersection B
A’	Complement of Set A

Sets Operation

The sets undergo various operation which includes their union of sets, the intersection of sets, the difference of sets, the complement of sets, and the Cartesian Product of Sets. We will learn them briefly below:

Union of Sets

Union of Sets basically refers to uniting two sets and writing their elements in a single set without repeating elements if common elements are present. The union of sets is given by $A \cup B$.
For Example if Set $A = \{2, 4\}$ and Set $B = \{4, 6\}$ then $A \cup B = \{2, 4\} \cup \{4, 6\} = \{2, 4, 6\}$

Intersection of Sets

Intersection of sets refers to finding the common elements between two sets. It is given by $A \cap B$. For Example if set $A = \{2, 4\}$ and $B = \{4, 6\}$ then $A \cap B = \{2, 4\} \cap \{4, 6\} = \{6\}$.

Difference of Sets

Difference of Sets refers to the deletion of common elements of two sets and writing down the remaining elements of two sets. It is represented as $A – B$. For Example if et $A = \{2, 4\}$ and $B = \{4, 6\}$ then $A – B = \{2, 6\}$

Complement of Set

Compliment of Set refers to the set of elements from the universal set excluding the elements of the set of which we are finding the compliment. It is given by A’. For Example, if we have to find out the complement of the set of Natural Numbers then it will include all the numbers in the set from the Real Numbers except the Natural Numbers. Here Real Number is the Universal set of Natural Numbers.

Cartesian product of Sets

Cartesian Product of Sets refers to the product between the elements of two sets in ordered pair. It is given as $A \times B$. For Example if set $A = \{2, 4\}$ and $B = \{4, 6\}$ then $A \times B = \{(2,4), (2,6), (4,4), (4,6)\}$.

2.1.7 Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

EXAMPLE 22 What do the statements $\forall x \in \mathbf{R}\,(x^2 \geq 0)$ and $\exists x \in \mathbf{Z}\,(x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

Set Theory Formulas

The set theory formulas are given for two kinds of sets overlapping and disjoint sets. Let's learn them separately

Given that two sets A and B are overlapping, the formulas are as follows:

n(A ∪ B)	n(A) + n(B) – n(A ∩ B)
n(A ∩ B)	n(A) + n(B) – n(A ∪ B)
n(A)	n(A ∪ B) + n(A ∩ B) – n(B)
n(B)	n(A ∪ B) + n(A ∩ B) – n(A)
n(A – B)	n(A ∪ B) – n(B)

$n(A \cup B)$	$n(A) + n(B) - n(A \cap B)$
$n(A - B)$	$n(A) - n(A \cap B)$
$n(A \cup B \cup C)$	$n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

If two sets A and B are disjoint sets

$n(A \cup B)$	$n(A) + n(B)$
$(A \cap B)$	Φ
$n(A - B)$	$n(A)$

Properties of Sets

The various properties followed by sets are tabulated below:

Property	**Expression**
Commutative Property	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative Property	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
Distributive Property	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Property	$A \cup \Phi = A$ $A \cap U = A$

Property	Expression
Complement Property	A ∪ A’ = A
Idempotent Property	A ∪ A = AA ∩ A = A

De Morgan’s Law

De Morgan’s Law is applicable in relating the union and intersection of two sets via their complements. There are two laws under De Morgan’s Law. Let’s learn them briefly

De Morgan’s Law of Union

De Morgan’s Law of Union states that the complement of the union of two sets is equal to the intersection of the complement of individual sets. Mathematically it can be expressed as

(A ∪ B)’ = A’ ∩ B’

De Morgan’s Law of Intersection

De Morgan’s Law of Intersection states that the complement of the intersection of two sets is equal to the union of the complement of individual sets. Mathematically it can be expressed as

(A ∩ B)’ = A’ ∪ B’

Proof of De-Morgan Laws:

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

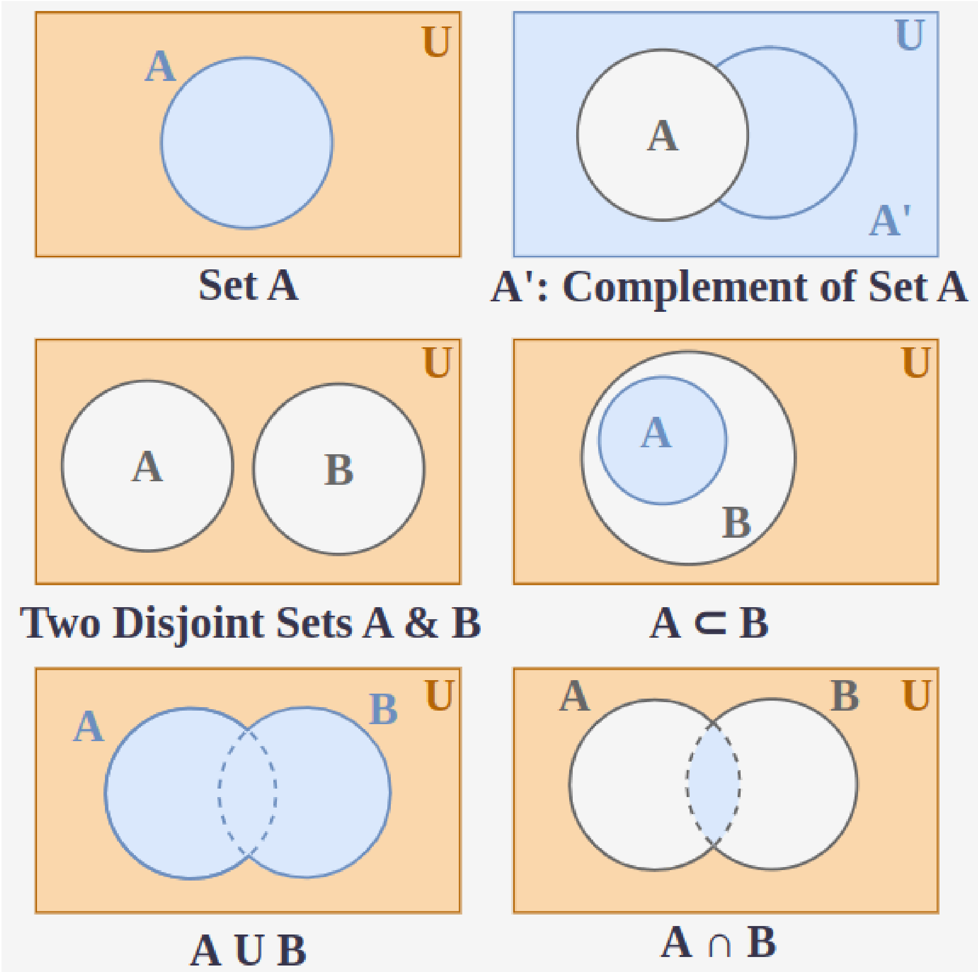
Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid.

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Venn Diagram

Venn Diagram is a technique for representing the relation between two sets with the help of circles, generally intersecting. For Example, two circles intersecting with each other with the common area merged into them represent the union of sets, and two intersecting circles with a common area highlighted represents the intersection of sets while two circles separated from each other represents the two disjoint sets. A rectangular box surrounding the circle represents the universal set. The Venn diagrams for various operations of sets are listed below:



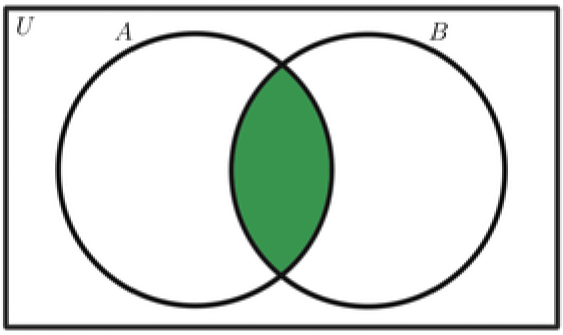
Operations on Sets

Intersection and Union of Sets

Intersection

The intersection of two sets A and B is a set that contains all the elements that are common to both A and B. Formally it is written as

In the following image, the shaded area is the intersection of sets A and B



$A \cap B$

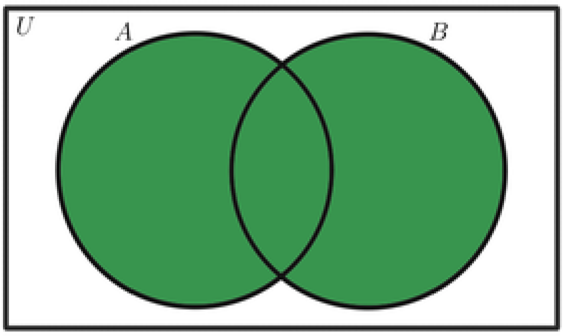
Example:
If A = {2, 3, 5, 7} and B = {1, 2, 3, 4, 5}
then the intersection of set A and B is the set A ∩ B = {2, 3, 5}

In this example 2, 3, and 5 are the only elements that belong to both sets A and B.

Union

Union of two sets A and B is a set that contains all the elements that are in A or in B or in both A and B. Formally it is written as

In the following image, the shaded area is the union of sets A and B.



$A \cup B$

Example:

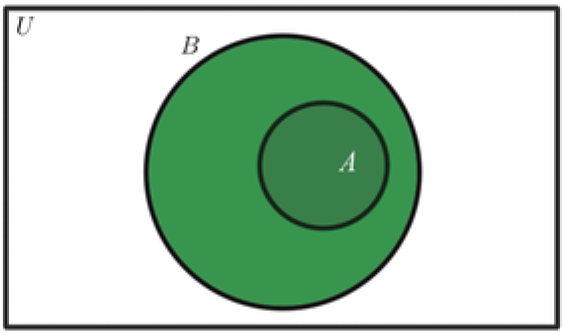
If A = {2, 4, 8} and B = {2, 6, 8}
then the union of A and B is the set A ∪ B = {2, 4, 6, 8}

In this example, 2, 4, 6, and 8 are the elements that are found in set A or in set B or in both sets A and B

Subset and Proper Subset

Subset

For two set A and B, A is a subset of B if every element in A is also in B. A can be equal to B. This is formally written as In the following image, set A is a subset of B



$A \subseteq B$

Example:
If A = {2, 4} and B = {1, 2, 3, 4, 5, 6, 7, 8}
then A is a subset of B

In this example, A is a subset of B, because all the elements in A are also in B

Notes:
1. An empty set (or null set) is a subset of every set.
Example:
∅ is a subset of the set {1, 2, 3, 4}

2. For a set A, the number of possible subsets is $2^{|A|}$. Where |A| = number of elements in A.

Example:
For the set C = {1, 2, 3}, there are 23 = 8 possible subsets
they are ∅, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, {1, 2, 3}

Proper subset (also called strict subset)

For two sets A and B, A is a proper subset of B, if A is a subset of B and A is not equal to B. Formally it is written as

Example:

For a set B = {1, 2, 3},

∅, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3} are all proper subsets of B

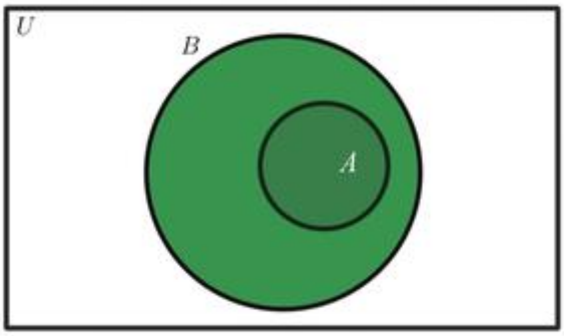
Note that {1, 2, 3} is not a proper subset of B, because they are equal

Superset and Proper Superset

Superset

For two sets A and B, if A is a subset of B then B is the superset of A. A can be equal to B. Formally it is denoted as

In the following image, set B is the superset of set A



$B \supseteq A$

Examples:

If A = {2, 4} and B = {1, 2, 3, 4, 5, 6, 7, 8}

then B is the superset of A, because A is a subset of B

If A = {11, 12} and B = {11, 12 } then B is the super set of A

Proper superset (also called strict superset)

For two sets A and B, if A is a subset of B and A is not equal to B, then B is the proper superset of A. Formally it is written as

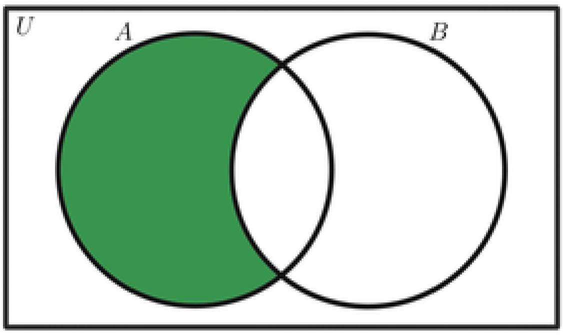
Examples:

- If A = {1, 2, 3} and B = {0, 1, 2, 3, 4, 5}then B is a proper superset of A, because A is a subset of B and A ≠ B

- If A = {2, 4, 6} and B = {2, 4, 6} then B is not a proper superset of A, because A = B

Relative Complement or Difference Between Sets

The relative complement or set difference of two sets A and B is the set containing all the elements that are in A but not in B. Formally this is written as Sometimes this is also written as A \ B. In the following image, the shaded area represents the difference set of set A and set B



$A - B$

Note: $A - B$ is equivalent to $A \cap B'$ i.e., $A - B = A \cap B'$

Example:

If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{2, 3, 5, 7\}$
then $A - B = \{1, 4, 6, 8, 9, 10\}$
and further $B - A = \emptyset$

Universal Set and Absolute Complement

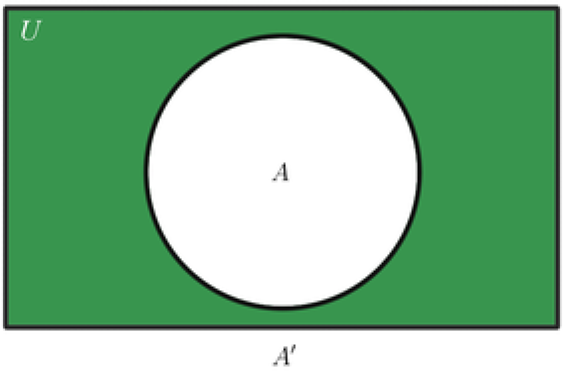
Universal set

A universal set is the set of all objects currently under consideration. It is usually denoted by the upper-case letter **U**. For example for a set of vowels, the universal set may be the set of alphabets.

Note: A set is always a subset of the universal set.

Absolute complement

The absolute complement of a set A is the set of all elements that are in U but not in A. It is denoted as A’. In the following image, the shaded area represents the complement of set A



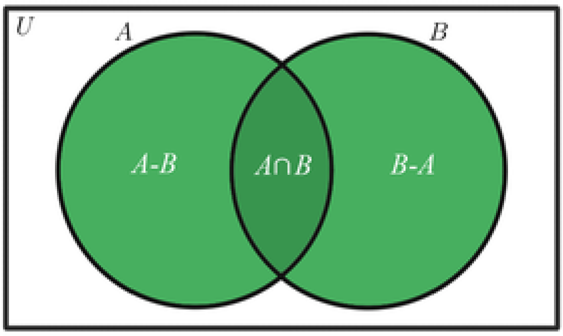
The absolute complement is sometimes just called complement.
Note: A’ is equivalent to U – A i.e., A’ = U – A
Example:
If U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 2, 3}
then A’ = {4, 5, 6, 7, 8, 9, 10} = U – A

Bringing the Set Operations Together

De Morgan’s laws

1. The complement of the union of two sets is equal to the intersection of their complements
   i.e., $(A \cup B)' = A' \cap B'$
2. The complement of the intersection of two sets is equal to the union of their complements
   i.e., $(A \cap B)' = A' \cup B'$

Formula for the Cardinality of Union and Intersection



$|A \cup B| = |A - B| + |A \cap B| + |B - A|$

The formula for the Cardinality of Union and Intersection is given below:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof:

We can write

$|A \cup B| = |A - B| + |A \cap B| + |B - A|$ —- by the sum of disjoint sets, refer to the Venn diagram above

$|A \cup B| = (|A| - |A \cap B|) + |A \cap B| + |B - A|$ —- Substitute $|A - B| = |A| - |A \cap B|$

$|A \cup B| = |A| + |B - A|$ —- Simplify

$|A \cup B| = |A| + |B| - |A \cap B|$ —- Substitute $|B - A| = |B| - |A \cap B|$)

Practical Problems on Union and Intersection of Two sets

Problem 1: There are 100 students in a class, 45 students said that they liked apples, and 30 of the students said that they liked both apples and oranges. Every student has to choose at least one of the two fruits. Find how many students like oranges.

Solution:

Let U = set of all students in the class

A = set of students that like apples

B = set of students that like oranges

Given:

$|A| = 45$

$|A \cap B| = 30$

$|U| = |A \cup B| = 100$ (because every student has to choose)

We need to find how many like oranges. i.e., $|B|$

The formula to be used is,

$|A \cup B| = |A| + |B| - |A \cap B|$ —-(i)

Subtract $|A| - |A \cap B|$ from both sides in (i) to get

$|A \cup B| - (|A| - |A \cap B|) = |B|$

or $|B| = |A \cup B| - (|A| - |A \cap B|)$

Substitute the given values and simplify,

$|B| = |A \cup B| - (|A| - |A \cap B|)$

$= 100 - (45 - 30)$

$= 85$

Thus the number of students that like oranges is 85.

Problem 2: There are a total of 120 students in a class. 70 of them study mathematics, 40 study science, and 10 students study both mathematics and science. Find the number of students who

i) Study mathematics but not science

ii) Study science but not mathematics

iii) Study mathematics or science

Solution:

Let,

U = set of all students in the class

M = set of students that study mathematics

S = set of students that study science

Our universal set here has 120 student i.e, |U| = 120

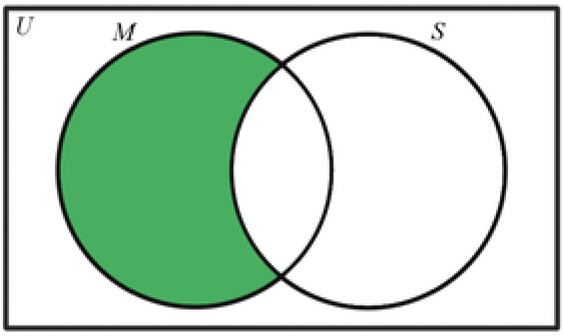
Given,

|M| = 70

|S| = 40

|M ∩ S| = 10 (number of students that study both mathematics and science)

i) Finding the number of students that study mathematics but not science. In the following image, the shaded area represents the set of students that study mathematics but not science.



We are required to find |M – S|

By the Venn diagram, we can see that |M – S| can be written as |M| – |M ∩ S|

thus,

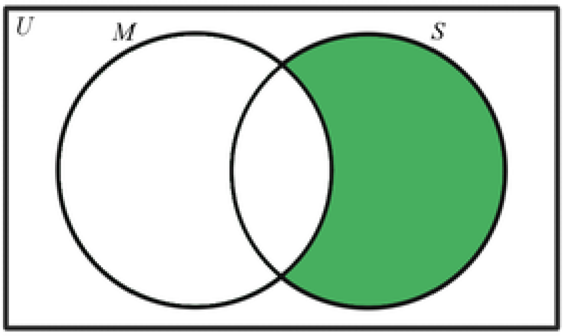
|M – S| = |M| – |M ∩ S|

= 70 – 10

= 60

Thus the number of students who study mathematics but not science is 60

ii) Finding the number of students that study science but not mathematics. In the following image, the shaded area represents the set of students that study science but not mathematics

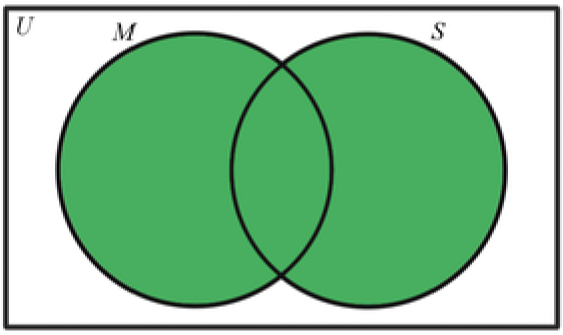


We are required to find |S – M|
By the Venn diagram, we can see that |S – M| can be written as |S| – |M ∩ S|
thus,

$$|S - M| = |S| - |M \cap S|$$
$$= 40 - 10$$
$$= 30$$

Thus the number of students who study science but not mathematics is 30

iii) Finding the number of students who study mathematics or science. In the following image, the shaded area represents the set of students that study mathematics or science.



We are required to find $|M \cup S|$

By using the formula, $|M \cup S| = |M| + |S| - |M \cap S|$

$$
\begin{aligned}
|M \cup S| &= |M| + |S| - |M \cap S| \\
&= 70 + 40 - 10 \\
&= 100
\end{aligned}
$$

Thus the number of students who study science or mathematics is 100.

Computer Representation of Sets

Method for storing elements using an arbitrary ordering of the elements of the universal set.Specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example1: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

i) What bit strings represent the subset of all odd integers in U?

The bit string that represents the set of odd integers in U, {1, 3, 5, 7, 9}, has a one bit in the First, third, fifth, seventh, and ninth positions. It is 10 1010 1010.

ii) What bit strings represent the subset of all even integers in U?

The bit string that represent the subset of even integers in U, {2, 4, 6, 8, 10}. It is 01 0101 0101.

iii) What bit strings represent the subset of integers not exceeding 5 in U?

The set of all integers in U that do not exceed 5, {1, 2, 3, 4, 5}, is represented by the String 11 1110 0000.

To find the bit string for the complement of a set from the bit string for that set, change each 1 to 0 and each 0 to 1.

Example 2: The bit string for the set {1, 3, 5, 7, 9} (with universal set {1, 2, 3, 4,5, 6, 7, 8, 9, 10}) is 10 1010 1010.

What is the bit string for the complement of this set?

The bit string for the complement of this set is obtained by replacing 0s with 1s.

This yields the string 01 0101 0101, which corresponds to the set {2, 4, 6, 8, 10}.
To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean Operations on the bit strings representing the two sets.

Example 3: The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010.
The bit string for the union of these sets is 11 1110 0000 ∨ 10 1010 1010 = 11 1110 1010, Which corresponds to the set {1, 2, 3, 4, 5, 7, 9}.If either of the bits in the ith position in the two strings is 1 (or both are 1), the bit in the ith position of the bit string of the union is 1. When both bits are 0, is 0. Hence, the bit string for union is the bitwise OR of the bit strings for the two sets.
The bit string for the intersection of these sets is11 1110 0000 ∧ 10 1010 1010 = 10 1010 0000, which corresponds to the set {1, 3, 5}. When the bits in the corresponding position in the two strings are both 1, the bit in the ith position of the bit string of the intersection is 1. When either of the two bits is 0 (or both are 0), is 0. Hence, the bit string for the intersection is the bitwise AND of the bit strings for the two sets.

MULTISETS

We recall that a set is a collection of distinct objects. There are many occasions, however, when we encounter collections of nondistinct objects. For example, consider the names of the students in a class. We might have two or more students who have the same name, and we might wish to talk about the collection of the names of the students. We define a multiset to be a collection of objects that are not necessarily distinct. Thus, **{a, a, a, b, b, c}**, **{a, a, a, a}**, **{a, b, c}**, and **{ }** are examples of multisets. The multiplicity of an element in a multiset is defined to be the number of times the element appears in the multiset. Thus, the multiplicity of the element a in the multiset **{a, a, a, c, d, d}** is **3**. The multiplicity of the element b is 0, the multiplicity if element c is 1, and the multiplicity of the element d is **2**. Not that sets are merely special instances of multisets in which the multiplicity of an element is either **0** or **1**.

The cardinality of a multiset is defined to be the cardinality of the set it corresponds to, assuming that the elements in the multiset are all distinct.

Let P and Q be two multi sets. The union of P and Q, denoted **P ∪ Q**, is a multiset such that the multiplicity of an element in **P ∪ Q** is equal to the maximum of the multiplicities of the element in P and in Q. thus, for **P = {a, a, a, c, d, d} and Q = {a, a, b, c, c**}

P ∪ Q = {a, a, a, b, c, c, d, d}

The intersection of P and Q, denoted **P ∩ Q**, is a multiset such that the multiplicity of an element in **P ∩ Q** is equal to the multiplicities of the element in P and in Q. Thus, for

P = {a, a, a, c, d, d} and **Q = {a, a, b, c, c}**

P ∩ Q = {a, a, c}

The difference of P and Q, denoted **P – Q**, is a multiset such that the multiplicity of the element in Q if the difference is positive, and is equal to 0 if the difference is 0 or is negative. For example, Let

P = {a, a, a, b, b, c, d, d, e} and **Q = {a, a, b, b, b, c, c, d, d, f}**. We have,

P – Q = {a, e}

Finally we define the sum of two multisets **P and Q,** denoted **P + Q,** to be a multiset such that the multiplicity of an element in **P + Q** is equal to the sum of the multiplicities of the element in **P** and in **Q**. note that there is no corresponding definition of the sum of two sets. For example, let **P = {a, a, b, c, c}** and **Q = {a, b, b, d}**. We have **P + Q = {a, a, a, b, b, b, c, c, d}**. As another example, let **R** be a multiset containing the account numbers of all the transactions on the next day. **R** and **S** are multisets because an account might have more than one transaction in a day. Thus **R +S** Is a combined record of the account numbers of the transactions in these two days?

FUNCTIONS

Functions are an important part of discrete mathematics. This article is all about functions, their types, and other details of functions. A function assigns exactly one element of a set to each element of the other set. Functions are the rules that assign one input to one output. The function can be represented as **f: A ⇢ B.** A is called the domain of the function and B is called the codomain function.

Functions:

- A function assigns exactly one element of one set to each element of other sets.
- A function is a rule that assigns each input exactly one output.
- A function f from **A to B** is an assignment of exactly one element of B to each element of A (where A and B are non-empty sets).
- A function f from set A to set B is represented as **f: A ⇢ B** where A is called the domain of f and B is called as codomain of f.
- If b is a unique element of B to element a of A assigned by function F then, it is written as **f(a) = b.**
- Function f maps A to B means f is a function from A to B i.e. f: **A ⇢ B**

Domain of a function:

- If f is a function from set **A** to set **B** then, A is called the domain of function f.
- The set of all inputs for a function is called its domain.

Codomain of a function:

- If f is a function from set A to set B then, B is called the codomain of function f.
- The set of all allowable outputs for a function is called its codomain.

Pre-image and Image of a function:

A function f: A ⇢ B such that for each a ∈ A, there exists a unique b ∈ B such that (a, b) ∈ R then, a is called the pre-image of f and b is called the image of f.

Types of function:

One-One function (or Injective Function):

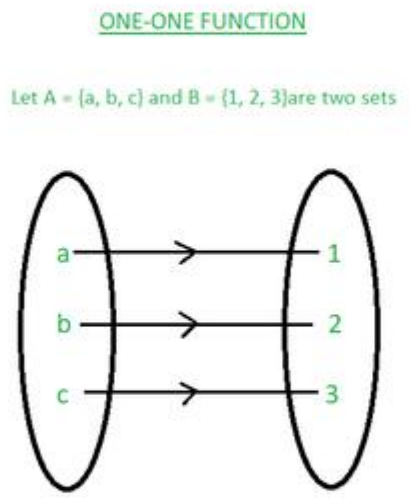
A function in which one element of the domain is connected to one element of the codomain.

A function f: A ⇢ B is said to be a one-one (injective) function if different elements of A have different images in B.

f: A ⇢ B is one-one

⇒ $a \neq b \Rightarrow f(a) \neq f(b)$ for all a, b ∈ A

⇒ $f(a) = f(b) \Rightarrow a = b$ for all a, b ∈ A



ONE-ONE FUNCTION

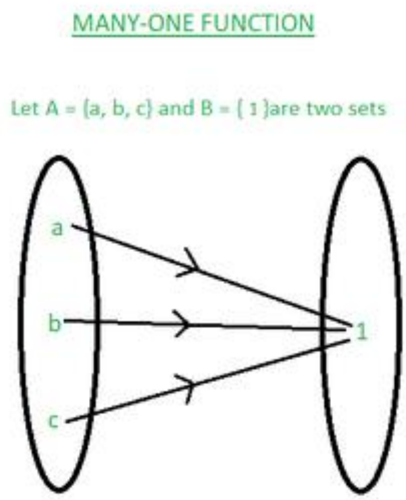
Many-One function:

A function f: A ⇢ B is said to be a many-one function if two or more elements of set A have the same image in B.

A function f: A ⇢ B is a many-one function if it is not a one-one function.

f: A ⇢ B is many-one

⇒ $a \neq b$ but $f(a) = f(b)$ for all a, b ∈ A

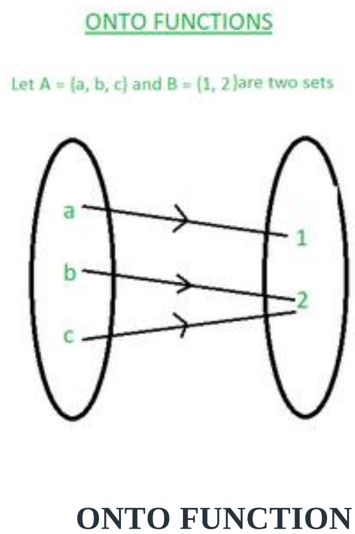


MANY-ONE FUNCTION

Onto function(or Surjective Function):

A function f: A -> B is said to be onto (surjective) function if every element of B is an image of some element of A i.e. f(A) = B or range of f is the codomain of f.
A function in which every element of the codomain has one pre-image.

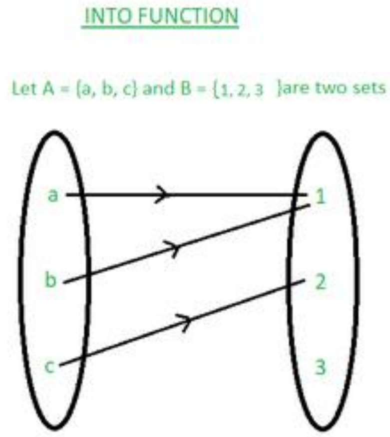
f: A ⇢ B is onto if for each b∈ B, there exists a∈ A such that f(a) = b.



ONTO FUNCTION

Into Function:

A function f: A ⇢ B is said to be an into a function if there exists an element in B with no pre-image in A.
A function f: A⇢ B is into function when it is not onto.



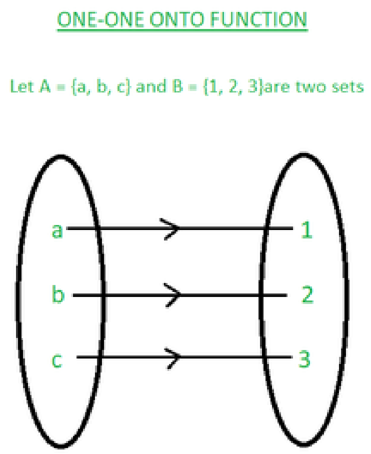
INTO FUNCTION

One-One Correspondent function(or Bijective Function or One-One Onto Function):

A function which is both one-one and onto (both injective and surjective) is called one-one correspondent(bijective) function.

f : A ⇢ B is one-one correspondent (bijective) if:

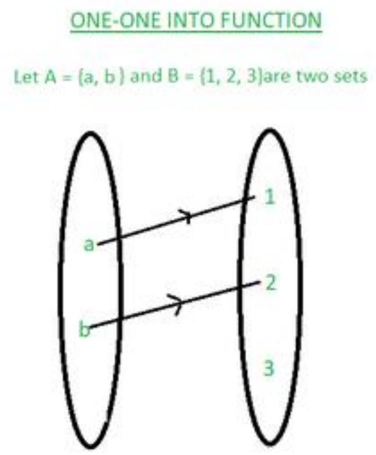
- **one-one i.e. f(a) = f(b) ⇒ a = b for all a, b ∈ A**
- **onto i.e. for each b ∈ B, there exists a ∈ A such that f(a) = b.**



ONE-ONE CORRESPONDENT FUNCTION

One-One Into function:

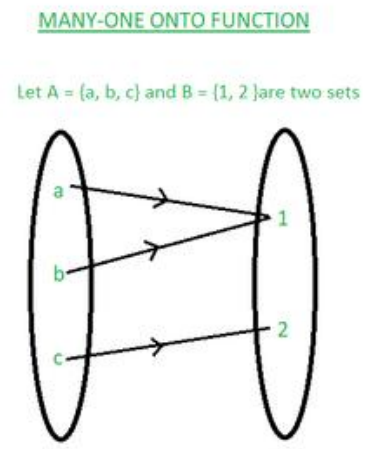
A function that is both one-one and into is called one-one into function.



ONE-ONE INTO FUNCTION

Many-one onto function:

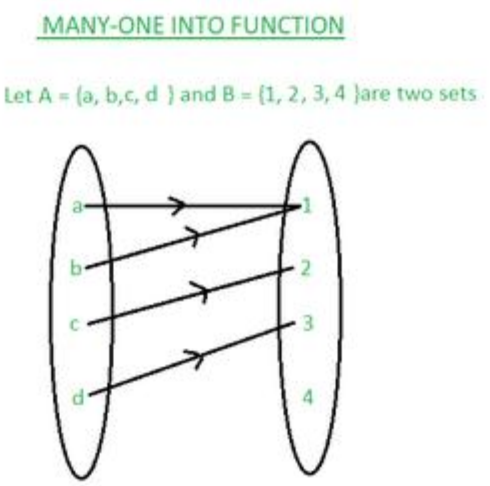
A function that is both many-one and onto is called many-one onto function.



MANY-ONE ONTO FUNCTION

Many-one into a function:

A function that is both many-one and into is called many-one into function.



MANY-ONE INTO FUNCTION

Inverse of a function:

Let $f: A \dashrightarrow B$ be a bijection then, a function $g: B \dashrightarrow A$ which associates each element $b \in B$ to a different element $a \in A$ such that $f(a) = b$ is called the inverse of f.

$\mathbf{f(a) = b \;\leftrightarrow\; g(b) = a}$

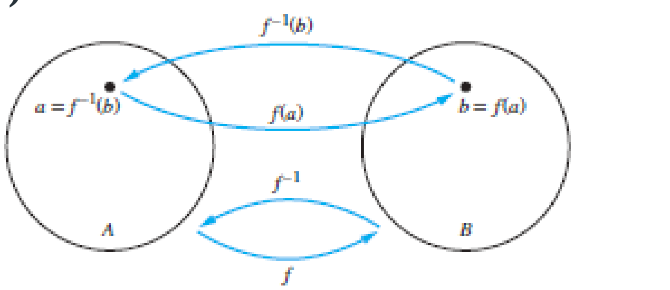


FIGURE 6 The function f^{-1} is the inverse of function f.

Composition of functions :-

Let f: A ⇢ B and g: B ⇢ C be two functions then, a function gof: A ⇢ C is defined by **(gof)(x) = g(f(x)), for all x ∈ A** is called the composition of f and g.

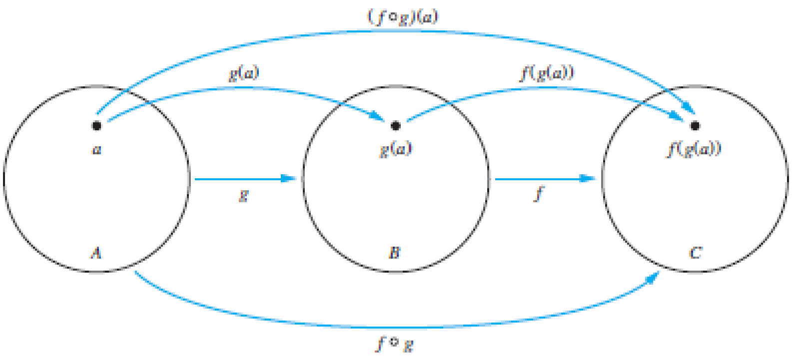


FIGURE 7 **The composition of the functions *f* and *g*.**

Note:

Let X and Y be two sets with m and n elements and a function is defined as f : X->Y then,

- Total number of functions = n^m
- Total number of one-one function = ${}^nP_m$
- Total number of onto functions = $n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - \ldots\ldots\ldots + (-1)^{n-1}\,{}^nC_{n-1}1^m$ if $m \geq n$.

For the composition of functions f and g be two functions :

- fog ≠ gof
- If f and g both are one-one function then fog is also one-one.
- If f and g both are onto function then fog is also onto.
- If f and fog both are one-one function then g is also one-one.
- If f and fog both are onto function then it is not necessary that g is also onto.
- $(fog)^{-1} = g^{-1}o\ f^{-1}$
- $f^{-1}o\ f = f^{-1}(f(a)) = f^{-1}(b) = a$
- $fof^{-1} = f(f^{-1}(b)) = f(a) = b$

  problem :Let *f* and *g* be the functions from the set of integers to the set of integers defined by
  $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of *f* and *g*? What is the compositionof *g* and *f* ?
  Solution*:* Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$
and
$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

Sample Questions:

Ques 1: Show that the function f : R ⇢ R, given by f(x) = 2x, is one-one and onto.

Sol: For one-one:

Let a, b ∈ R such that f(a) = f(b) then,

$f(a) = f(b)$

$\Rightarrow 2a = 2b$

$\Rightarrow a = b$

Therefore, f: R ⇢ R is one-one.

For onto:

Let p be any real number in R (co-domain).

$f(x) = p$

$\Rightarrow 2x = p$

$\Rightarrow x = p/2$

$p/2 \in R$ for $p \in R$ such that $f(p/2) = 2(p/2) = p$

For each p∈ R (codomain) there exists x = p/2 ∈ R (domain) such that f(x) = y

For each element in codomain has its pre-image in domain.

So, f: R ⇢ R is onto.

Since f: R ⇢ R is both one-one and onto.

f : R ⇢ R is one-one correspondent (bijective function).

Ques 2: Let $f : R \dashrightarrow R$; $f(x) = \cos x$ and $g : R \dashrightarrow R$; $g(x) = x^3$. Find fog and gof.

Sol: Since the range of f is a subset of the domain of g and the range of g is a subset of the domain of f. So, fog and gof both exist.

gof (x) = g(f(x)) = g(cos x) = $(\cos x)^3 = \cos^3 x$

fog (x) = f(g(x)) = $f(x^3) = \cos x^3$

Ques 3: If $f : Q \dashrightarrow Q$ is given by $f(x) = x^2$, then find $f^{-1}(16)$.

Sol:

Let $f^{-1}(16) = x$

$f(x) = 16$

$x^2 = 16$

$x = \pm 4$

$f^{-1}(16) = \{-4, 4\}$

Ques 4 :- If $f : R \dashrightarrow R$; $f(x) = 2x + 7$ is a bijective function then, find the inverse of f.

Sol: Let x ∈ R (domain), y ∈ R (codomain) such that f(a) = b

$f(x) = y$

$\Rightarrow 2x + 7 = y$

$\Rightarrow x = (y - 7)/2$

$\Rightarrow f^{-1}(y) = (y - 7)/2$

Thus, $f^{-1} : R \dashrightarrow R$ is defined as $f^{-1}(x) = (x - 7)/2$ for all $x \in R$.

Ques 5: If $f : A \dashrightarrow B$ and $|A| = 5$ and $|B| = 3$ then find total number of functions.

Sol: Total number of functions = $3^5 = 243$

Graph of function

We can associate a set of pairs in $A \times B$ to each function from A to B. This set of pairs is called the graph of the function and is often displayed pictorially to aid in understanding the behaviour of the function.

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a)=b\}$.

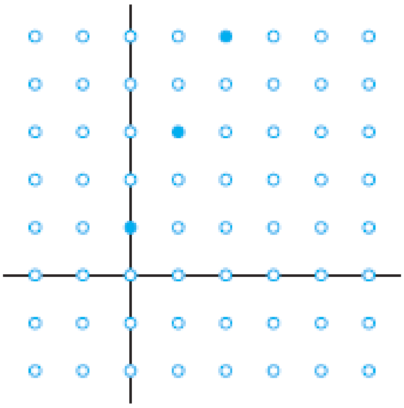


FIGURE 8 The graph of $f(n) = 2n + 1$ from Z to Z.

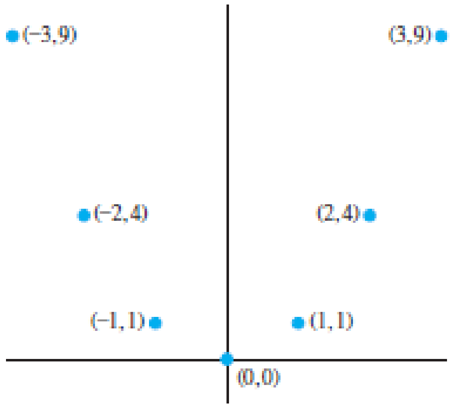


FIGURE 9 The graph of $f(x) = x^2$ from Z to Z.

Some important function

The *floor* function, denoted

$f(x)=[x]$

is the largest integer less than or equal to x

The *ceiling* function, denoted

$f(x)=[x]$

is the smallest integer greater than or equal to x

ex; $[3.5]=4 [3.5]=3$

$[-1.5]=-1$ $[-1.5]=-2$

Floor and ceiling function

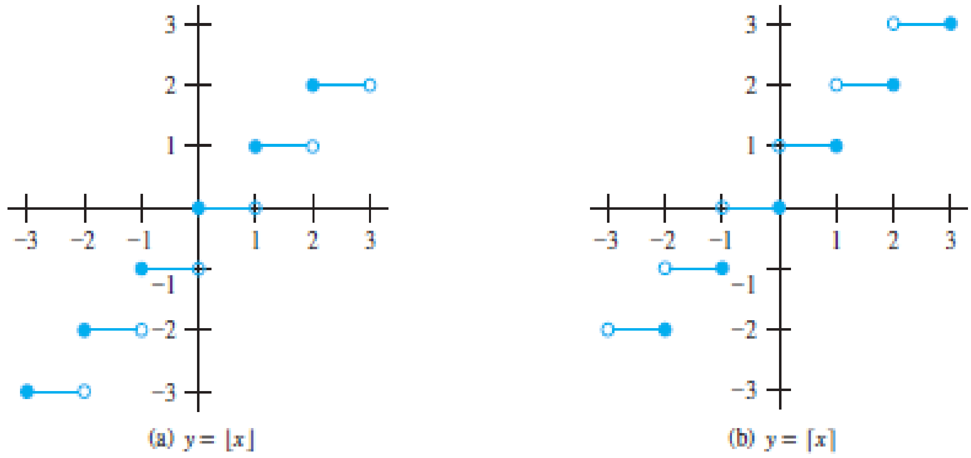


FIGURE 10 Graphs of the (a) floor and (b) ceiling functions.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving properties of function

Ex: Prove that x is a real number, then

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$!

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \le \varepsilon < 1$.

Case 1: $\varepsilon < 1/2$

$2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \le 2\varepsilon < 1$.

$\lfloor x + 1/2 \rfloor = n$, since x + . = $n + (1/2 + \varepsilon)$ and $0 \le ½ + \varepsilon < 1$.

Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq .1/2$
$2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since
$0 \leq 2\varepsilon - 1 < 1$.
$\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Factorial function

Definition: $f: N \to Z+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.
$f(n) = 1 \cdot 2 \cdots (n - 1) \cdot n$, $f(0) = 0! = 1$
Examples:
$f(1) = 1! = 1$
$f(2) = 2! = 1 \cdot 2 = 2$
$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$
$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000$.
Stirling's Formula
$n! \sim \sqrt{2\pi n}(n/e)n$
$f(n) \sim g(n) = \lim_{n\to\infty} f(n)/g(n) = 1$

PARTIAL FUNCTION

Definition: A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B.
The sets A and B are called the domain and codomain of f, respectively. We day that f is undefined for elements in A that are not inthe domain of definition of f. !When the domain of definition of f equals A, we say that f isa total function.
Example: $f: N \to R$ where $f(n) = \sqrt{n}$ is a partial function from Z to R where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

SEQUENCES AND SUMMATIONS

Introduction

- Sequences are ordered lists of elements.
- 1, 2, 3, 5, 8!
- 1, 3, 9, 27, 81, …….

Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.

We will introduce the terminology to represent sequences and sums of the terms in the sequences.

SEQUENCES

Definition: A sequence is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.

● The notation an is used to denote the image of the integer n. We can think of an as the equivalent of f(n) where f is a function from {0,1,2,…..} to S. We call an a term of the sequence

ex:

Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

GEOMETRIC PROGRESSION

Definition: A geometric progression is a sequence of the form :where the initial term a and the common ratio r are real Numbers

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Examples:

Let $a = 1$ and $r = -1$. Then:
$\{b_n\}=\{b_1,b_2,b_3,b_4\ldots\ldots\}=\{-1,1,-1,1\ldots\ldots\}$
Let $a = 2$ and $r = 5$. Then:
$\{C_n\}=\{c_0,c_1,c_2,c_3,c_4\ldots\ldots\}=\{-1,1,-1,1\ldots\ldots\}$
Let $a = 6$ and $r = 1/3$. Then:
$\{d_n\}=\{d_1,d_2,d_3,d_4\ldots\ldots\}=\{6,2,2/3,2/9.2/27\ldots\ldots\}$

Arithmetic progression

An *arithmetic progression* is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

Examples:

1. Let $a = -1$ and $d = 4$:
$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:
$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and d = 2:
$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).!

- Sequences of characters or bits are important in computer science.
- The empty string is represented by λ.
- The string abcde has length 5.

Recurrence Relations

Definition: A recurrence relation for the sequence {an} is an equation that expresses an in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, for all integers n with n ≥ n0, where n0 is a nonnegative integer.

- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
- The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Definition 4 A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence. We will explain this alternative terminology in Chapter 5.)

EXAMPLE 5 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. ◀

Questions on Recurrence

Relations

Example 1: Let {an} be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,…. and suppose that $a_0 = 2$.

What are a_1 , a_2 and a_3

[Here a0 = 2 is the initial condition.]

Solution: We see from the recurrence relation that

a1 = a0 + 3 = 2 + 3 = 5

a2 = 5 + 3 = 8

a3 = 8 + 3 = 11

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \cdots$. and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that!

$a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Fibonacci Sequence

Definition: Define the Fibonacci sequence, f_0 ,f_1 ,f_2,…, by:

● Initial Conditions: $f_0 = 0$, $f_1 = 1$

● Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2 ,f_3 ,f_4 , f_5 and f_6 .

Answer:

$f_2 = f_1 + f_0 = 1 + 0 = 1$,

$f_3 = f_2 + f_1 = 1 + 1 = 2$,

$f_4 = f_3 + f_2 = 2 + 1 = 3$,

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$f_6 = f_5 + f_4 = 5 + 3 = 8$.

Solving Recurrence Relations

● Finding a formula for the nth term of the sequence generated by a recurrence relation is called solving the recurrence relation.

● Such a formula is called a **closed formula**.

● Here we illustrate by example the method of iteration in which we need to guess the formula.

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,…. and suppose that $a_1 = 2$.

$a_2 = 2 + 3$

$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$

$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$

.$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,…. and suppose that $a_1 = 2$.

$a_n = a_{n-1} + 3$

$= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$

$= (a_{n-3} + 3)+ 3 \cdot 2 = a_{n-3} + 3 \cdot 3$

.$= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1$

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$

with the initial condition $P_0 = 10{,}000$

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$

with the initial condition P0 = 10,000

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

$P_n = (1.11)P_{n-1} = (1.11)^nP_0 = (1.11)^n\ 10{,}000$

$P_n = (1.11)^n\ 10{,}000$

$P_{30} = (1.11)^{30}\ 10{,}000 = \$228{,}992.97$

Special Integer Sequences (opt)

● Given a few terms of a sequence, try to identify the sequence.Conjecture a formula, recurrence relation, or some other rule.

● Some questions to ask?

● Are there repeated terms of the same value?

● Can you obtain a term from the previous term by adding an amount or multiplying by an amount?

● Can you obtain a term by combining the previous terms insome way?!

● Are they cycles among the terms?

● Do the terms match those of a well known sequence

Questions on Special Integer Sequences (opt)

Example 1: Find formulae for the sequences with the following first five terms: 1, ., ., 1/8, 1/16

Solution: Note that the denominators are powers of 2. The sequence with an = $1/2^n$ is a possible match. This is a geometric progression with a = 1 and r = 1/2

Example 2: Consider 1,3,5,7,9

Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is an = 2n + 1. This is an arithmetic progression with a =1 and d = 2.

Example 3: 1, -1, 1, -1,1

Solution: The terms alternate between 1 and -1. A possible sequence is an = $(-1)^n$. This is a geometric progression with a = 1 and r = −1.

Some useful sequences

TABLE 1 Some Useful Sequences.	
nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …

Guessing Sequences (optional)

Example: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: Note the ratio of each term to the previous approximates 3. So now compare with the sequence 3^n . We notice that the nth term is 2 less than thecorresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$

Integer Sequences (optional)

● Integer sequences appear in a wide range of contexts the number of ways to order n discrete objects the number of moves needed to solve the Tower of Hanoi puzzle with n disks and the number of rabbits on an island after n months

● Integer sequences are useful in many fields such as biology, engineering, chemistry and physics.
● On-Line Encyclopedia of Integer Sequences (OESIS) contains over 200,000 sequences. Began by Neil Stone in the 1960s (printed form). Now found at http://oeis.org/ Spuzzle.html
Here are three interesting sequences to try from the OESIS site. To solve each puzzle, find a rule that determines the terms of the sequence.Guess the rules for forming for the following sequences:
● 2, 3, 3, 5, 10, 13, 39, 43, 172, 177,
● Hint: Think of adding and multiplying by numbers to generate this sequence.0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
● Hint: Think of the English names for the numbers representing the position in the sequence and the Roman Numerals for the same number. 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
● Hint: Think of the English names for numbers, and whether or not they have the letter‘e.’The answers and many more can be found at http://oeis.org/Spuzzle.html

SUMMATIONS

Sum of the terms from the sequence

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$. We use the notation

n

The notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \cdots + a_n.$$

The variable **j** is called the index of summation. It runs through all the integers starting with its lower limit **m** and ending with its upper limit **n**.
More generally for a set S:

What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2\\ &= 1 + 4 + 9 + 16 + 25\\ &= 55.\end{aligned}$$

What is the value of $\sum_{k=4}^{8} (-1)^k$?

Solution: We have

$$\begin{aligned}\sum_{k=4}^{8} (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8\\ &= 1 + (-1) + 1 + (-1) + 1\\ &= 1.\end{aligned}$$

Sometimes it is useful to shift the index of summation in a sum. This is often done when two sums need to be added but their indices of summation do not match. When shifting an indexof summation, it is important to make the appropriate changes in the corresponding summand.This is illustrated by Example

Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

Product notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$

- The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

represents $a_m \times a_{m+1} \times \cdots \times a_n$

Geometric progression

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Proof: Let

$$S_n = \sum_{j=0}^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned}
rS_n &= r\sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\
&= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\
&= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j+1 \\
&= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a) && \text{removing } k = n+1 \text{ term and adding } k = 0 \text{ term} \\
&= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula}
\end{aligned}$$

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$.

EXAMPLE 21 Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned} \sum_{i=1}^{4} \sum_{j=1}^{3} ij &= \sum_{i=1}^{4} (i + 2i + 3i) \\ &= \sum_{i=1}^{4} 6i \\ &= 6 + 12 + 18 + 24 = 60. \end{aligned}$$

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values $f(s)$, for all members s of S.

EXAMPLE 22 What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Because $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of s for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

Some important summations

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Find $\sum_{k=50}^{100} k^2$.

Solution: First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2 (and proved in Exercise 38), we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

SOME INFINITE SERIES Although most of the summations in this book are finite sums, infinite series are important in some parts of discrete mathematics. Infinite series are usually studied in a course in calculus and even the definition of these series requires the use of calculus, but sometimes they arise in discrete mathematics, because discrete mathematics deals with infinite collections of discrete elements. In particular, in our future studies in discrete mathematics, we will find the closed forms for the infinite series in Examples 24 and 25 to be quite useful.

:XAMPLE 24 (*Requires calculus*) Let x be a real number with $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$.

Extra Examples

Solution: By Theorem 1 with $a = 1$ and $r = x$ we see that $\sum_{n=0}^{k} x^n = \frac{x^{k+1} - 1}{x - 1}$. Because $|x| < 1$, x^{k+1} approaches 0 as k approaches infinity. It follows that

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.$$

We can produce new summation formulae by differentiating or integrating existing formulae.

:XAMPLE 25 (*Requires calculus*) Differentiating both sides of the equation

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x},$$

from Example 24 we find that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}.$$

(This differentiation is valid for $|x| < 1$ by a theorem about infinite series.)

MATRICES

Introduction

Matrices are used throughout discrete mathematics to express relationships between elements in sets. In subsequent chapters we will use matrices in a wide variety of models. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models. This section reviews matrix arithmetic that will be used in these algorithms.

Definition 1 A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an *m* × *n* matrix. The plural of matrix is *matrices*. A matrix with the same number of rows as columns is called *square*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix.

Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The ith *row* of $\mathbf{A}$ is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The jth *column* of $\mathbf{A}$ is the $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}.$$

The (i, j)th *element* or *entry* of $\mathbf{A}$ is the element a_{ij}, that is, the number in the ith row and jth column of $\mathbf{A}$. A convenient shorthand notation for expressing the matrix $\mathbf{A}$ is to write $\mathbf{A} = [a_{ij}]$, which indicates that $\mathbf{A}$ is the matrix with its (i, j)th element equal to a_{ij}.

Matrix Arithmetic

The basic operations of matrix arithmetic will now be discussed, beginning with a definition of matrix addition.

Definition 3 Let A = [aij] and **B = [bij] be m × n** matrices. The *sum* of **A** and **B**, denoted by **A** + **B**, is the *m* × *n* matrix that has **aij + bij as it's (i, j)** th element. In other words, **A** + **B = [aij + bij].**

The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, because such matrices will not both have entries in some of their positions

We have $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$.

Matrix multiplication

Let **A** be an *m* × *k* matrix and **B** be a *k* × *n* matrix. The *product* of **A** and **B**, denoted by **AB**, is the *m* × *n* matrix with its (*i*, *j*)th entry equal to the sum of the products of the corresponding elements from the *i*th row of **A** and the *j*th column of **B**. In other words, if **AB** = [*cij*], then

cij = *ai*1*b*1*j* + *ai*2*b*2*j* +···+ *aikbkj*.

The product of two matrices is not defined when the number of columns in the first matrix and the number of rows in the second matrix are not the same.

We now give some examples of matrix products

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find **AB** if it is defined.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

FIGURE 1 **The product of $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$.**

Solution: Because **A** is a 4 × 3 matrix ***Extra*** and **B** is a 3 × 2 matrix, the product **AB** is defined and is a

Examples 4 × 2 matrix. To find the elements of **AB**, the corresponding elements of the rows of **A** and the columns of **B** are first multiplied and then these products are added. For instance, the element in the (3, 1)th position of **AB** is the sum of the products of the corresponding elements of the third row of **A** and the first column of **B**; namely, $3 \cdot 2 + 1 \cdot 1 + 0 \cdot 3 = 7$. When all the elements of **AB** are computed, we see that

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}.$$

Although matrix multiplication is associative, as can easily be proved using the associativityof addition and multiplication of real numbers, matrix multiplication is *not* commutative. Thatis, if **A** and **B** are two matrices, it is not necessarily true that **AB** and **BA** are the same.

In fact, itmay be that only one of these two products is defined. For instance, if **A** is 2 × 3 and **B** is 3 × 4, then **AB** is defined and is 2 × 4; however, **BA** is not defined, because it is impossible to multiply 3 × 4 matrix and a 2 × 3 matrix.

In general, suppose that **A** is an $m \times n$ matrix and **B** is an $r \times s$ matrix. Then **AB** is defined only when $n = r$ and **BA** is defined only when $s = m$. Moreover, even when **AB** and **BA** are both defined, they will not be the same size unless $m = n = r = s$. Hence, if both **AB** and **BA** are defined and are the same size, then both **A** and **B** must be square and of the same size.

Furthermore, even with **A** and **B** both $n \times n$ matrices, **AB** and **BA** are not necessarily equal, as

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

Solution: We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$.

Transpose of matrix

Transpose of a matrix is a matrix that is obtained by swapping the rows and columns of the given matrix or vice versa, i.e., for the given matrix the elements in rows are interchanged with the elements in columns. For any given matrix A its transpose is denoted as A^t, or A^T. Let, **A** is a matrix of order **m × n** then $\mathbf{A^t}$ be the transpose of matrix A with order **n × m**,

$A = [a_{(ij)}]_{m \times n}$

$A^t = [a_{(ji)}]_{n \times m}$

here i, j present the position of a matrix element, row- and column-wise, respectively, such that, $1 \leq i \leq m$ and $1 \leq j \leq n$.

If $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$

and $j = 1, 2, \ldots, m$

Example: For any given matrix A of order 2 × 3 its transpose is?

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 7 & 0 \end{bmatrix}$$

Solution:

Transpose of A

$$A^t = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ 3 & 0 \end{bmatrix}$$

Order of A^t is ***3 × 2***

Matrices that do not change when their rows and columns are interchanged are often important

A square matrix **A** is called *symmetric* if **A** = **A***t*. Thus, **A** = [*aij*] is symmetric if *aij* = *aji* for all *i* and *j* with 1 ≤ *i* ≤ *n* and 1 ≤ *j* ≤ *n*.

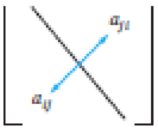


FIGURE 2 A symmetric matrix.

Note that a matrix is symmetric if and only if it is square and it is symmetric with respect to its main diagonal (which consists of entries that are in the *i*th row and *i*th column for some *i*). This symmetry is displayed in Figure 2.

EXAMPLE 6 The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric.

Power matrix

Definition: The identity matrix of order n is the m n matrix **I**n = [δij], where δij = 1 if i = j and δij = 0 if i≠j.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when **A** is an $m \times n$ matrix, we have

$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$.

Powers of square matrices can be defined because matrix multiplication is associative.

When **A** is an $n \times n$ matrix, we have

$$\mathbf{A}^0 = \mathbf{I}_n, \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA} \cdots \mathbf{A}}_{r \text{ times}}.$$

ZERO-ONE MATRICES

Definition: A matrix all of whose entries are either 0or 1 is called a *zero-one matrix*

Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean Operations $\wedge$ and $\vee$, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero–one matrices. Then the *join* of **A** and **B** is the zero–one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of **A** and **B** is denoted by $\mathbf{A} \vee \mathbf{B}$. The *meet* of **A** and **B** is the zero–one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of **A** and **B** is denoted by $\mathbf{A} \wedge \mathbf{B}$.

Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

BOOLEAN PRODUCT OF ZERO-ONE MATRIX

Let **A = [*aij*]** be an ***m*** × ***k*** zero‑one matrix and **B = [*bij*] be a *k*** × ***n*** zero‑one matrix. Then the *Boolean product* of A and B, denoted by A ⊙B, is the ***m*** × ***n*** matrix with (*i*, *j*)th entry ***cij***

where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

the Boolean product of **A** and **B** is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation ∨ and with multiplication replaced with the operation ∧. We give an example of the Boolean products of matrices.

Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

BOOLEAN POWER OF ZERO-ONE MATRIX

Boolean powers of a square zero–one matrix. These powers will be used in our subsequent studies of paths in graphs, which are used to model such things as communications paths in computer networks.

Let $\mathbf{A}$ be a square zero–one matrix and let r be a positive integer. The rth *Boolean power* of $\mathbf{A}$ is the Boolean product of r factors of $\mathbf{A}$. The rth Boolean product of $\mathbf{A}$ is denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$. ◀