Unit-1:Determinants & Matrix

Introduction and Examples

DEFINITION: A matrix is defined as an ordered rectangular array of numbers. They can be used to represent systems of linear equations, as will be explained below.Here are a example $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -5 \\ 3 & -5 & 6 \end{bmatrix}$

Square

A **square** matrix has the same number of rows as columns.

$$\begin{bmatrix} 2 & 0 \\ 1 & 8 \end{bmatrix}$$

A square matrix (2 rows, 2 columns)

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \\ 3 & 0 & 7 \end{bmatrix}$$

Also a square matrix (3 rows, 3 columns)

Identity Matrix

An **Identity Matrix** has **1**s on the main diagonal and **0**s everywhere else:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×3 Identity Matrix

- It is square (same number of rows as columns)
- It can be large or small (2×2, 100×100, ... whatever)
- Its symbol is the capital letter **I**

It is the matrix equivalent of the number "1", when we multiply with it the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Diagonal Matrix

A diagonal matrix has zero anywhere not on the main diagonal:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A diagonal matrix

Scalar Matrix

A scalar matrix has all main diagonal entries the same, with zero everywhere else:

Triangular Matrix

Lower triangular is when all entries above the main diagonal are zero:

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & 6 & -3 \end{bmatrix}$$

A lower triangular matrix

Upper triangular is when all entries below the main diagonal are zero:

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 4 & 11 \\ 0 & 0 & 5 \end{bmatrix}$$

An upper triangular matrix

Zero Matrix (Null Matrix)

Zeros just everywhere:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrix

Matrix Addition and Subtraction

DEFINITION: Two matrices A and B can be added or subtracted if and only if their dimensions are the same (i.e. both matrices have the same number of rows and columns. Take:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

Addition

If A and B above are matrices of the same type then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of adding A and B together.

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 2 & 0 & 5 \end{pmatrix}$$

Subtraction

If A and B are matrices of the same type then the subtraction is found by subtracting the corresponding elements $a_{ij} - b_{ij}$.

Here is an example of subtracting matrices.

$$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Matrix Multiplication

DEFINITION: When the number of columns of the first matrix is the same as the number of rows in the second matrix then matrix multiplication can be performed.

Here is an example of matrix multiplication for two 2×2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{pmatrix}$$

Here is an example of matrix multiplication for two 3×3 matrices.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{pmatrix}$$

Note: That A×B is not the same as B×A

Transpose of Matrices

DEFINITION: The transpose of a matrix is found by exchanging rows for columns i.e. Matrix A = (a_{ij}) and the transpose of A is:

A^T = (a_{ji}) where j is the column number and i is the row number of matrix A.

For example, the transpose of a matrix would be:

$$A = \begin{pmatrix} 5 & 2 & 3 \\ 4 & 7 & 1 \\ 8 & 5 & 9 \end{pmatrix} \quad A^T = \begin{pmatrix} 5 & 4 & 8 \\ 2 & 7 & 5 \\ 3 & 1 & 9 \end{pmatrix}$$

In the case of a square matrix (m = n), the transpose can be used to check if a matrix is symmetric. For a symmetric matrix A = A^T.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = A^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = A$$

The Determinant of a Matrix

DEFINITION: Determinants play an important role in finding the inverse of a matrix and also in solving systems of linear equations. In the following we assume we have a square matrix (m = n). The determinant of a matrix A will be denoted by det(A) or |A|. Firstly the determinant of a 2×2 and 3×3 matrix will be introduced, then the n×n case will be shown.

Determinant of a 2×2 matrix

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of a 3×3 matrix

The determinant of a 3×3 matrix is a little more tricky and is found as follows (for this case assume A is an arbitrary 3×3 matrix A, where the elements are given below).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} -2 & 3 & -1 \\ 5 & -1 & 4 \\ 4 & -8 & 2 \end{vmatrix} = -2\begin{vmatrix} -1 & 4 \\ -8 & 2 \end{vmatrix} -5\begin{vmatrix} 3 & -1 \\ -8 & 2 \end{vmatrix}$$

$$+4\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix}$$

$$= -2[(-1)(2) - (-8)(4)] - 5[(3)(2)$$
$$-(-8)(-1)] +4[(3)(4) \; -(-1)(-1)]$$

$$= -2(30) - 5(-2) + 4(11)$$

$$= -60 + 10 + 44$$

$$= -6$$

Here, we are **expanding by the first column.** We can do the expansion by using the first row and we will get the same result.

The Inverse of a Matrix

DEFINITION: Assuming we have a square matrix A, which is non-singular (i.e. det(A) does not equal zero), then there exists an n×n matrix A^{-1} which is called the inverse of A, such that this property holds:

$AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

The inverse of a 2×2 matrix

Take for example a arbitury 2×2 Matrix A whose determinant (ad − bc) is not equal to zero.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a,b,c,d are numbers, The inverse is:

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse Matrix Method

DEFINITION: The inverse matrix method uses the inverse of a matrix to help solve a system of equations, such like the above Ax = b. By pre-multiplying both sides of this equation by A^{-1} gives:

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$

or alternatively

$$x = A^{-1}b$$

So by calculating the inverse of the matrix and multiplying this by the vector b we can find the solution to the system of equations directly. And from earlier we found that the inverse is given by

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

From the above it is clear that the existence of a solution depends on the value of the determinant of A. There are three cases:

1. If the det(A) does not equal zero then solutions exist using

2. If the det(A) is zero and b=0 then the solution will be not be unique or does not exist.
3. If the det(A) is zero and b=0 then the solution can be x = 0 but as with 2. is not unique or does not exist.

Looking at two equations we might have that

$$ax + by = c$$
$$dx + ey = f$$

Written in matrix form would look like

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

and by rearranging we would get that the solution would look like

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$$

Cofactors

The 2 × 2 determinant

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

is called the **cofactor** of a_1 for the 3 × 3 determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The cofactor is formed from the elements that are not in the same row as a_1 and not in the same column as a_1.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Similarly, the determinant

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

is called the **cofactor** of a_2. It is formed from the elements not in the same row as a_2 and not in the same column as a_2.

We continue the pattern for the cofactor of a_3.

Cramer's Rule to Solve 3 × 3 Systems of Linear Equations

We can solve the general system of equations,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

by using the determinants:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\Delta}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example 4

Solve, using Cramer's Rule:

$$2x + 3y + z = 2$$

$$-x + 2y + 3z = -1$$

$$-3x - 3y + z = 0$$

$$x = \frac{\begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 0 & -3 & 1 \end{vmatrix}}{\Delta}$$

$$y = \frac{\begin{vmatrix} 2 & 2 & 1 \\ -1 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} 2 & 3 & 2 \\ -1 & 2 & -1 \\ -3 & -3 & 0 \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ -3 & -3 & 1 \end{vmatrix} = 2(11) + 1(6) - 3(7)$$
$$= 7$$

So

$$x = \frac{2(11) + 1(6) + 0}{7} = \frac{28}{7} = 4$$

$$y = \frac{2(-1) + 1(2) - 3(7)}{7} = -\frac{21}{7} = -3$$

$$z = \frac{2(-3) + 1(6) - 3(-7)}{7} = \frac{21}{7} = 3$$

Checking solutions:

[1] $2(4) + 3(\text{-}3) + 3 = 2$ OK

[2] $-(4) + 2(\text{-}3) + 3(3) = \text{-}1$ OK

[3] $-3(4) - 3(-3) + 3 = 0$ OK

So the solution is $(4, -3, 3)$.

Rank of a Matrix

The rank of a matrix with m rows and n columns is a number **r** with the following properties:

- r is less than or equal to the smallest number out of m and n.
- r is equal to the order of the greatest minor of the matrix which is not 0.

Determining the Rank of a Matrix

- We pick an element of the matrix which is not 0.
- We calculate the order 2 minors which contain that element until we find a minor which is not 0.
- If every order 2 minor is 0, then the rank of the matrix is 1.
- If there is any order 2 minor which is not 0, we calculate the order 3 minors which contain the previous minor until we find one which is not 0.

- If every order 3 minor is 0, then the rank of the matrix is 2.
- If there is any order 3 minor which is not 0, we calculate the order 4 minors until we find one which is not 0.
- We keep doing this until we get minors of an order equal to the smallest number out of the number of rows and the number of columns.

Example

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 5 \end{pmatrix}$$

The matrix has 2 rows and 3 columns, so the greatest possible value of its rank is 2. We pick any element which is not 0.

$$\begin{pmatrix} \color{red}{1} & 2 & 4 \\ 3 & 6 & 5 \end{pmatrix}$$

We form an order 2 minor containing 1.

$$\begin{pmatrix} \color{red}{1} & \color{red}{2} & 4 \\ \color{red}{3} & \color{red}{6} & 5 \end{pmatrix}$$

We calculate this minor.

$$\begin{vmatrix} \color{red}{1} & \color{red}{2} \\ \color{red}{3} & \color{red}{6} \end{vmatrix} = 6 - 6 = 0$$

We form another order 3 minor containing 1. $A =$

$$\begin{pmatrix} \color{blue}{1} & 2 & \color{blue}{4} \\ \color{blue}{3} & 6 & \color{blue}{5} \end{pmatrix}$$

We calculate this minor.

$$\begin{vmatrix} \color{blue}{1} & \color{blue}{4} \\ \color{blue}{3} & \color{blue}{5} \end{vmatrix} = 5 - 12 = -7 \neq 0.$$

The rank is 2.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We pick an element which is not 0.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We calculate order 2 minors containing this element.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \text{ (because it has two equal rows)}$$

Every other order 2 minor is 0 because it's the same as the others. In this case, the rank of the matrix is 1.

Eigen Values and Eigen Vectors

Eigen vector of a matrix A is a vector represented by a matrix X such that when X is multiplied with matrix A, then the direction of the resultant matrix remains same as vector X.

Mathematically, above statement can be represented as:

$AX = \lambda X$

where A is any arbitrary matrix, λ are eigen values and X is an eigen vector corresponding to each eigen value.

Here, we can see that AX is parallel to X. So, X is an eigen vector.

Method to find eigen vectors and eigen values of any square matrix A
We know that,

$AX = \lambda X$

$\Rightarrow AX - \lambda X = 0$

$\Rightarrow (A - \lambda I) X = 0$ …..(1)

Above condition will be true only if $(A - \lambda I)$ is singular. That means,

$|A - \lambda I| = 0$ …..(2)

(2) is known as characteristic equation of the matrix.
null

The roots of the characteristic equation are the eigen values of the matrix A.

Now, to find the eigen vectors, we simply put each eigen value into (1) and solve it by Gaussian elimination, that is, convert the augmented matrix $(A - \lambda I) = 0$ to row echelon form and solve the linear system of equations thus obtained.

Some important properties of eigen values

- Eigen values of real symmetric and hermitian matrices are real
- Eigen values of real skew symmetric and skew hermitian matrices are either pure imaginary or zero
- Eigen values of unitary and orthogonal matrices are of unit modulus $|\lambda| = 1$

- If $\lambda_{1,} \lambda_2 \ldots\ldots\lambda_n$ are the eigen values of A, then $k\lambda_1$, $k\lambda_2 \ldots\ldots k\lambda_n$ are eigen values of kA
- If $\lambda_{1,} \lambda_2 \ldots\ldots\lambda_n$ are the eigen values of A, then $1/\lambda_1$, $1/\lambda_2 \ldots\ldots 1/\lambda_n$ are eigen values of A^{-1}
- If $\lambda_{1,} \lambda_2 \ldots\ldots\lambda_n$ are the eigen values of A, then λ_1^k, $\lambda_2^k \ldots\ldots\lambda_n^k$ are eigen values of A^k
  null
- Eigen values of A = Eigen Values of A^T (Transpose)
- Sum of Eigen Values = Trace of A (Sum of diagonal elements of A)
- Product of Eigen Values = |A|
- Maximum number of distinct eigen values of A = Size of A
- If A and B are two matrices of same order then, Eigen values of AB = Eigen values of BA

Note –Eigenvalues and eigenvectors are only for square matrices.

Eigenvectors are *by definition nonzero*. Eigenvalues may be equal to zero.

EXAMPLE 1: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

SOLUTION:

- In such problems, we first find the **eigenvalues** of the matrix.

FINDING EIGENVALUES

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A, namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 **identity matrix**.

- Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}.$$

Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

- Calculate $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\ &= (1-\lambda)\left((-5-\lambda)(4-\lambda) - (3)(-6)\right) + 3(3(4-\lambda) - 3\times 6) + 3(3\times(-6) - (-5-\lambda)6) \\ &= (1-\lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) \\ &= (1-\lambda)(-2 + \lambda + \lambda^2) + 3(-6 - 3\lambda) + 3(12 + 6\lambda) \\ &= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda \\ &= 16 + 12\lambda - \lambda^3. \end{aligned}$$

- Therefore

$$\det(A - \lambda I) = -\lambda^3 + 12\lambda + 16.$$

REQUIRED: To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

$$\lambda^3 - 12\lambda - 16 = 0. \tag{1}$$

  * Look for **integer** valued solutions.

* Such solutions **divide** the **constant** term (-16). The list of possible integer solutions is
$$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$
* Taking $\lambda = 4$, we find that $4^3 - 12.4 - 16 = 0$.
* Now factor out $\lambda - 4$:
$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$
* Solving $\lambda^2 + 4\lambda + 4$ by formula[1] gives
$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.4}}{2} = \frac{-4 \pm 0}{2},$$
and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the **characteristic equation**.)

FINDING EIGENVECTORS

- Once the **eigenvalues** of a matrix (A) have been found, we can find the **eigenvectors** by Gaussian Elimination.
- **STEP 1**: For each eigenvalue λ, we have
$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$
where x is the **eigenvector** associated with **eigenvalue** λ.
- **STEP 2**: Find $\mathbf{x}$ by Gaussian elimination. That is, convert the augmented matrix
$$\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$$
to row echelon form, and solve the resulting linear system by back substitution.

We find the **eigenvectors** associated with each of the **eigenvalues**

- **Case 1:** $\lambda = 4$
  - We must find vectors $\mathbf{x}$ which satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

[1]To find the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ (with $a \neq 0$) first compute $\Delta = b^2 - 4ac$, then if $\Delta \geq 0$ the roots exist and are equal to $x = \frac{-b-\sqrt{\Delta}}{2a}$ and $x = \frac{-b+\sqrt{\Delta}}{2a}$.

- First, form the matrix $A - 4I$:

$$A - 4I = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}.$$

- Construct the augmented matrix $\left(A - \lambda I \vdots \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{array}{c} \begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \xrightarrow{R1 \to -1/3 \times R3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow{R2 \to -1/12 \times R2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow{R3 \to R3 + 12 \times R2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \\
\xrightarrow{R1 \to R1 - R2} \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \end{array}$$

- Rewriting this augmented matrix as a linear system gives

$$\begin{aligned} x_1 - 1/2x_3 &= 0 \\ x_2 - 1/2x_3 &= 0 \end{aligned}$$

So the eigenvector $\mathbf{x}$ is given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = \frac{x_3}{2} \\ x_2 = \frac{x_3}{2} \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

For any real number $x_3 \neq 0$. Those are the **eigenvectors of A associated with the eigenvalue** $\lambda = 4$.

- **Case 2**: $\lambda = -2$

  - We seek vectors $\mathbf{x}$ for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
  - Form the matrix $A - (-2)I = A + 2I$

$$A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}.$$

- Now we construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{array}{c} \begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \end{array} \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \xrightarrow{R1 \to 1/3 \times R3} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

$$\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

- When this augmented matrix is rewritten as a linear system, we obtain

$$x_1 + x_2 - x_3 = 0,$$

so the eigenvectors $\mathbf{x}$ associated with the eigenvalue $\lambda = -2$ are given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

- Thus

$$\mathbf{x} = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for any } x_2, x_3 \in \mathbb{R} \backslash \{0\}$$

are the **eigenvectors of A associated with the eigenvalue** $\lambda = -2$.

------------END---------

B.C.A study

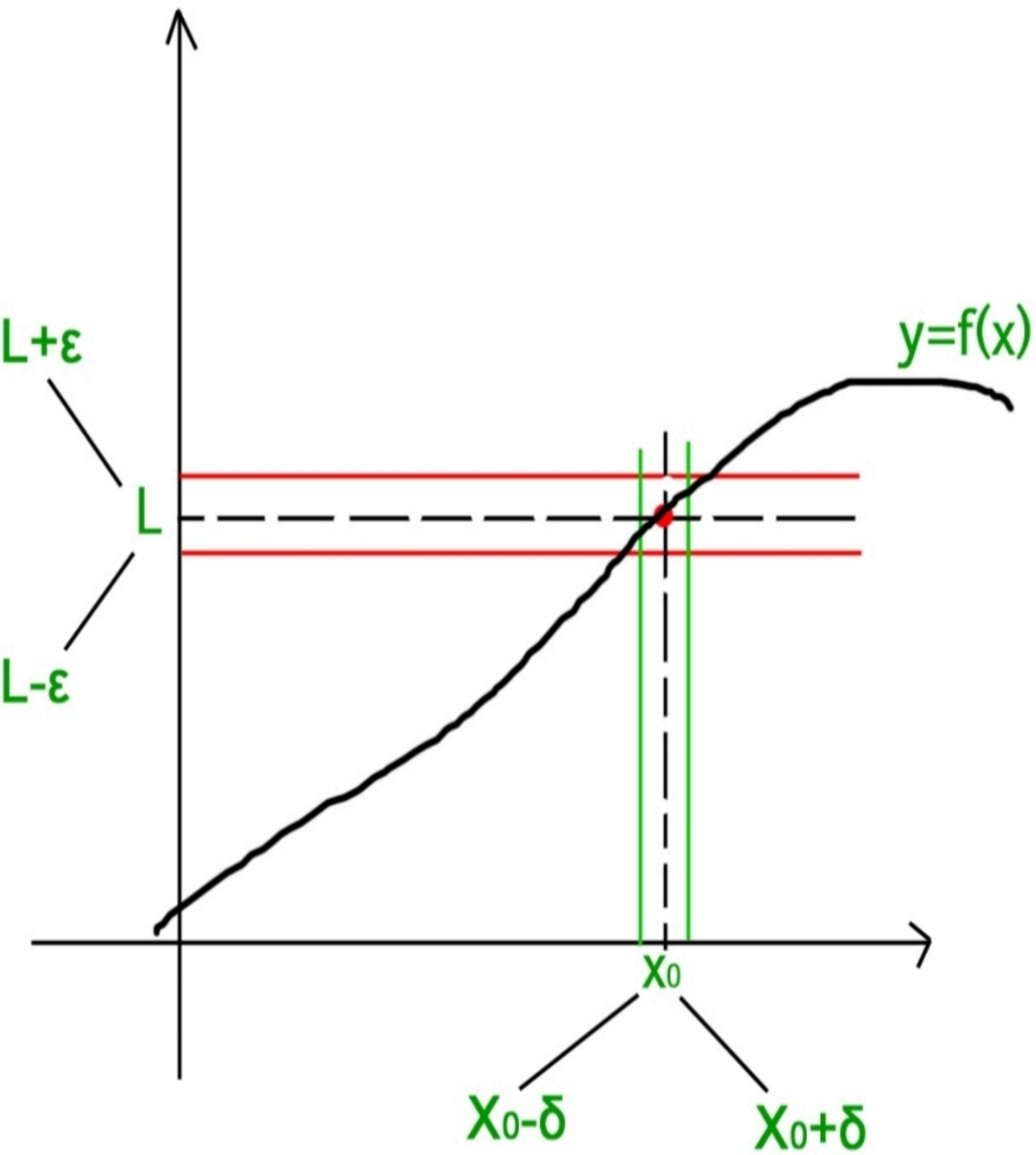
Unit-2:Limit and Continuity

1. Limits –

For a function $f(x)$ the limit of the function at a point $x = a$ is the value the function achieves at a point which is very close to $x = a$.

Formally,
Let $f(x)$ be a function defined over some interval containing $x = a$, except that it may not be defined at that point.
We say that, $L = \lim_{x \to a} f(x)$ if there is a number δ for every number ϵ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

The concept of limit is explained graphically in the following image –



As is clear from the above figure, the limit can be approached from either sides of the number line i.e. the limit can be defined in terms of a number less that a or in terms of a number greater than a. Using this criteria there are two types of limits –

Left Hand Limit – If the limit is defined in terms of a number which is less than a then the limit is said to be the left hand limit. It is denoted as $x \rightarrow a^-$ which is equivalent to $x = a - h$ where $h > 0$ and $h \rightarrow 0$.

Right Hand Limit – If the limit is defined in terms of a number which is greater than a then the limit is said to be the right hand limit. It is denoted as $x \rightarrow a^+$ which is equivalent to $x = a + h$ where $h > 0$ and $h \rightarrow 0$.

Existence of Limit – The limit of a function $f(x)$ at $x = a$ exists only when its left hand limit and right hand limit exist and are equal and have a finite value i.e.

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

Some Common Limits –

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \cos x = 1$
- $\lim_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$
- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \to \infty} (1 + \frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
- $\lim_{x \to 0} \frac{(a^x - 1)}{x} = \ln a$
- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$
- $\lim_{x \to \infty} x^{\frac{1}{x}} = 1$

Properties of Limit

- The limit of a function is represented as f(x) reaches L as x tends to limit a, such that; $\lim_{x \to a} f(x) = L$
- The limit of the sum of two functions is equal to the sum of their limits, such that: $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- The limit of any constant function is a constant term, such that, $\lim_{x \to a} C = C$
- The limit of product of the constant and function is equal to the product of constant and the limit of the function, such that: $\lim_{x \to a} m\, f(x) = m \lim_{x \to a} f(x)$
- Quotient Rule: $\lim_{x \to a}[f(x)/g(x)] = \lim_{x \to a} f(x)/\lim_{x \to a} g(x)$; if $\lim_{x \to a} g(x) \neq 0$

L'Hospital Rule –

If the given limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ i.e. both $f(x)$ and $g(x)$ are 0 or both $f(x)$ and $g(x)$ are ∞, then the limit can be solved by **L'Hospital Rule**.

If the limit is of the form described above, then the L'Hospital Rule says that –

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where $f'(x)$ and $g'(x)$ obtained by differentiating $f(x)$ and $g(x)$.

If after differentitating, the form still exists, then the rule can be applied continuously until the form is changed.

- **Example 1** – Evaluate
$$\lim_{x \to 0} \frac{x\cos(x) - \sin(x)}{x^2 \sin(x)}$$

- **Solution –** The limit is of the form 0, Using L'Hospital Rule and differentiating numerator and denominator

$$= \lim_{x \to 0} \frac{\cos(x) - x\sin(x) - \cos(x)}{x^2\cos(x) + 2x\sin(x)}$$

$$= \lim_{x \to 0} \frac{-\sin(x)}{x\cos(x) + 2\sin(x)}$$

$$= \lim_{x \to 0} \frac{\frac{-\sin(x)}{x}}{\cos(x) + \frac{2\sin(x)}{x}}$$

$$= \frac{-1}{1+2*1}$$

$$= \frac{-1}{3}$$

2. Continuity –

A function is said to be continuous over a range if it's graph is a single unbroken curve.

Formally,

A real valued function $f(x)$ is said to be continuous at a point $x = x_\circ$ in the domain if –

$\lim_{x \to x_\circ} f(x)$ exists and is equal to $f(x_\circ)$.

If a function $f(x)$ is continuous at $x = x_\circ$ then-

$\lim_{x \to x_\circ^+} f(x) = \lim_{x \to x_\circ^-} f(x) = \lim_{x \to x_\circ} f(x)$

Functions that are not continuous are said to be discontinuous.

- **Example 1 –** For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2), & \text{if x} \leq 0 \\ \text{4x+1}, & \text{otherwise} \end{cases}$$

continuous at $x = 0$?

- **Solution –** For the function to be continuous the left hand limit, right hand limit and the value of the function at that point must be equal.
  Value of function at $x = 0$
  $f(0) = \lambda * (0 - 2) = -2\lambda$
  Right hand limit-
  $= \lim_{x \to 0^+} 4x + 1$
  $= 1$
  RHL equals value of function at 0-
  $-2\lambda = 1$
  $\lambda = \frac{-1}{2}$

- **Example 2 –** Find all points of discontinuity of the function f defined by –
  $f(x) = |x| - |x - 1|$.
- **Solution –** The possible points of discontinuity are $x = 0, 1$ since the

sign of the modulus changes at these points.

For continuity at $x = 0$,

LHL-

$$= \lim_{x\to 0^-} |x| - |x-1|$$

$$= \lim_{x\to 0^-} -x - (-(x-1))$$

$$= \lim_{x\to 0^-} -x + x - 1$$

$$= -1$$

RHL

$$= \lim_{x\to 0^+} |x| - |x-1|$$

$$= \lim_{x\to 0^+} x - (-(x-1))$$

$$= \lim_{x\to 0^+} x + x - 1$$

$$= -1$$

Value of $f(x)$ at $x = 0$,

$f(0) = 0 - |0 - 1| = -1$

Since LHL = RHL = $f(0)$, the function is continuous at $x = 0$

For continuity at $x = 1$,

LHL–

$= \lim_{x \to 1^-} |x| - |x - 1|$

$= \lim_{x \to 1^-} x - (-(x - 1))$

$= \lim_{x \to 1^-} x + x - 1$

$= 1$

RHL

$= \lim_{x \to 1^+} |x| - |x - 1|$

$= \lim_{x \to 1^+} x - (x - 1)$

$= \lim_{x \to 1^+} x - x + 1$

$= 1$

Value of $f(x)$ at $x = 1$,

$f(0) = 1 - |1 - 1| = 1$

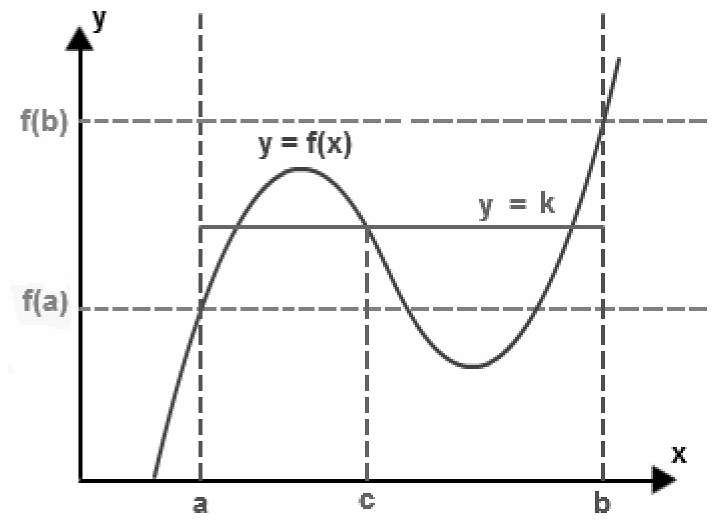
Since LHL = RHL = $f(1)$, the function is continuous at $x = 1$

There are three basic types of discontinuities:

1. **Removable (point) discontinuity** – the graph has a hole at a single x-value. Imagine you're walking down the road, and someone has removed a manhole cover (Careful! Don't fall in!). This function will satisfy condition #2 (limit exists) but fail condition #3 (limit does not equal function value).
2. **Infinite discontinuity** – the function goes toward positive or negative infinity. Imagine a road getting closer and closer to a river with no bridge to the other side
3. **Jump discontinuity** – the graph jumps from one place to another. Imagine a superhero going for a walk: he reaches a dead end and, because he can, flies to another road.

Intermediate Value Theorem Statement

Intermediate value theorem states that if "f" be a continuous function over a closed interval [a, b] with its domain having values f(a) and f(b) at the endpoints of the interval, then the function takes any value between the values f(a) and f(b) at a point inside the interval. This theorem is explained in two different ways:



Statement 1:

If k is a value between f(a) and f(b), i.e.

either $f(a) < k < f(b)$ or $f(a) > k > f(b)$

then there exists at least a number c within a to b i.e. $c \in (a, b)$ in such a way that $f(c) = k$

Statement 2:

The set of images of function in interval [a, b], containing [f(a), f(b)] or [f(b), f(a)], i.e.

either $f([a, b]) \supseteq [f(a), f(b)]$ or $f([a, b]) \supseteq [f(b), f(a)]$

Theorem Explanation:

The statement of intermediate value theorem seems to be complicated. But it can be understood in simpler words. Let us consider the above diagram, there is a continuous function f with endpoints a and b, then the height of the point "a" and "b" would be "f(a)" and "f(b)".

If we pick a height k between these heights f(a) and f(b), then according to this theorem, this line must intersect the function f at some point (say c), and this point must lie between a and b.

An intermediate value theorem, if c = 0, then it is referred to as Bolzano's **theorem**.

Intermediate Theorem Proof

We are going to prove the first case of the first statement of the intermediate value theorem since the proof of the second one is similar.

We will prove this theorem by the use of completeness property of real numbers.The proof of "$f(a) < k < f(b)$" is given below:

Let us assume that A is the set of all the values of x in the interval [a, b], in such a way that $f(x) \le k$.

Here A is supposed to be a non-empty set as it has an element "a" and also A is bounded above by the value "b".

Thus, by completeness property, we have that, "c" be the lowest value which is greater than or equal to each element of A. Hence, we can say that $f(c) = k$.

Given that f is continuous. Then let us consider a $\varepsilon > 0$, there exists "a $\delta > 0$" such that

$| f(x) – f(c) | < \varepsilon$ for every $| x – c | < \delta$. This gives us

$f(x) – \varepsilon < f(c) < f(x) + \varepsilon$

For each x lying within $c – \delta$ and $c + \delta$. So, we have values of x lying between c and $c - \delta$, contained in A, such that :

$f(c) < (f(x) + \varepsilon) \le (k + \varepsilon)$ —–– (1)

Similarly, values of x between c and $c + \delta$ that are not contained in A, such that

$f(c) > (f(x) – \varepsilon) > (k − \varepsilon)$ —––(2)

Combining both the inequality relations, obtain

$k – \varepsilon < f(c) < k + \varepsilon$

For every $\varepsilon > 0$

Hence, the theorem is proved.

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Unit-3:Differentiation

Differentiation

Differentiation allows us to find rates of change. For example, it allows us to find the rate of change of velocity with respect to time (which is acceleration). It also allows us to find the rate of change of x with respect to y, which on a graph of y against x is the gradient of the curve. There are a number of simple rules which can be used to allow us to differentiate many functions easily.

If y = some function of x (in other words if y is equal to an expression containing numbers and x's), then the **derivative** of y (with respect to x) is written dy/dx, pronounced "dee y by dee x" .

Derivatives are defined as the varying rate of change of a function with respect to an independent variable. The derivative is primarily used when there is some varying quantity, and the rate of change is not constant

a function is denoted as y=f(x), the derivative is indicated by the following notations.

1. **D(y) or D[f(x)]** is called Euler's notation.
2. **dy/dx** is called Leibniz's notation.
3. **F′(x)** is called Lagrange's notation.

Differentiation Formulas List

Differentiation Formulas List

In all the formulas below, f' means $\frac{d(f(x))}{dx} = f'(x)$ and g' means $\frac{d(g(x))}{dx}$ = $g'(x)$. Both f and g are the functions of x and differentiated with respect to x. We can also represent **dy/dx = D_x y**. Some of the general differentiation formulas are;

1. Power Rule: (d/dx) **(x^n)** = nx^{n-1}
2. Derivative of a constant, a: (d/dx) **(a)** = 0
3. Derivative of a constant multiplied

4. Sum Rule: (d/dx) **(f ± g)** = f' ± g'
5. Product Rule: (d/dx) **(fg)**= fg' + gf'
6. Quotient Rule: $\frac{d}{dx}\left(\frac{f}{g}\right)$ = $\frac{gf'-fg'}{g^2}$

Differentiation Formulas for Trigonometric Functions

Trigonometry is the concept of relation between angles and sides of triangles. Here, we have 6 main ratios, such as, sine, cosine, tangent, cotangent, secant and cosecant. You must have learned about basic trigonometric formulas based on these ratios. Now let us see, the formulas for derivative of trigonometric functions.

1. $\frac{d}{dx}(sin\ x) = cosx$
2. $\frac{d}{dx}(cos\ x) =- sinx$
3. $\frac{d}{dx}(tan\ x) = sec^2x$
4. $\frac{d}{dx}(cot\ x = -csc^2x$
5. $\frac{d}{dx}(sec\ x) = secxtanx$
6. $\frac{d}{dx}(csc\ x) = -cscxcotx$
7. $\frac{d}{dx}(sinh\ x) = coshx$
8. $\frac{d}{dx}(cosh\ x) = sinhx$
9. $\frac{d}{dx}(tanh\ x) = sech^2x$
10. $\frac{d}{dx}(coth\ x) = -csch^2x$

11. $\frac{d}{dx}(sech\ x) = -sechxtanhx$
12. $\frac{d}{dx}(csch\ x) = -cschx.cothx$

Differentiation Formulas for Inverse Trigonometric Functions

Inverse trigonometry functions are the inverse of trigonemetric ratios. Let us see the formulas for derivative of inverse trigonometric functions.

1. $\frac{d}{dx}(sin^{-1}\ x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(cos^{-1}\ x) = -\frac{1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(tan^{-1}\ x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(cot^{-1}\ x) = -\frac{1}{1+x^2}$
5. $\frac{d}{dx}(sec^{-1}\ x) = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}(csc^{-1}\ x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Other Differentiation Formulas

1. $\frac{d}{dx}(a^x) = a^x lna$
2. $\frac{d}{dx}(e^x) = e^x$
3. $\frac{d}{dx}(log_a\ x) = \frac{1}{(ln\ a)x}$
4. $\frac{d}{dx}(ln\ x) = 1/x$
5. Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}$

Rolle’s Theorem

Rolle’s theorem is one of the foundational theorems in differential calculus. It is a special case of, and in fact is equivalent to, the mean value theorem , which in turn is an essential ingredient in the proof of the fundamental theorem of calculus.

Rolle’s Theorem

Suppose f(x) be a function satisfying three conditions:

1) f(x) is continuous in the closed interval $a \le x \le b$

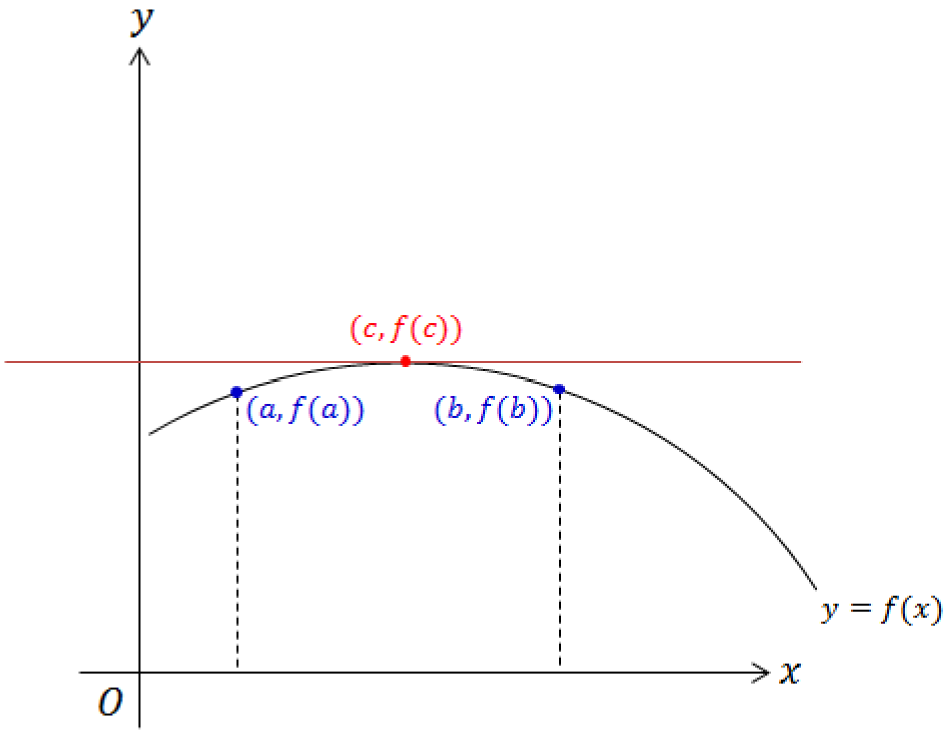
2) f(x) is differentiable in the open interval $a < x < b$

3) f(a) = f(b)

Then according to Rolle’s Theorem, there exists **at least one** point ‘c’ in the open interval (a, b) such that:

f ʻ (c) = 0

A graphical demonstration of this will help our understanding; actually, you’ll feel that it’s very apparent:



Imgur

In the figure above, we can set any two points as \(a,f(a)) and \(b,f(b)) as long as f(a)=f(b) and the function is differentiable within the interval (a,b). Then, of course, there has to be a point in between where f′=0,which is the red point in the diagram. Now let's take a look at the mathematical proof of this theorem.

We divide it into two cases:

(1) f(x) is a constant function.

If f(x)f(x) is a constant function, then f′=0 for the whole interval. Then, of course, there exists a c such that f′(c)=0 within the interval (a,b).

(2) f(x) is not a constant function.

When f(x) is not a constant function but is continuous within the interval [a,b],according to the extreme value theorem f(x) must have a maximum function value and minimum function value within the interval [a,b].Since f(x)is not a constant function, at least one of the extrema must exist within the interval (a,b).

(2)-1

If $f(x)$ has its maximum function value $f(c)$ at $x = c \in (a, b)$, then for a real number h whose absolute value is small enough that $a < c + h < b$, it follows that

$$f(c+h) - f(c) \leq 0.$$

Hence we have

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0, \quad \lim_{h \to 0^+} \frac{f(c+h)}{h}$$

Hence we have

$$\frac{h) - f(c)}{h} \geq 0, \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

Since $f(x)$ is differentiable in the interval (a, b), according to the squeeze theorem we have

$$0 \leq \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h)}{h}$$

$$=$$

$$\operatorname{im}_{\to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\Rightarrow f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= 0.$$

(2)-2

If $f(x)$ has its minimum function value $f(c)$ at $x = c \in (a, b)$, then for a real number h whose absolute value is small enough that $a < c + h < b$, it follows that

$$f(c+h) - f(c) \geq 0.$$

Hence we have

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \leq 0, \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \geq 0.$$

Since $f(x)$ is differentiable in the interval (a, b), according to the squeeze theorem we have

$$0 \leq \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h)}{h}$$

$$=$$

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\Rightarrow f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= 0.$$

EXAMPLE

When

$$f(x) = x^2 - 2x + 1,$$

show that $f'(x) = 0$ has at least one root in the interval $0 < x < 2$ using Rolle's theorem.

Observe that $f(x) = x^2 - 2x + 1$ is continuous in the interval $[0, 2]$ and differentiable in $(0, 2)$. $\qquad (1)$

The function values of $f(x)$ at $x = 0, 2$ are

$$\begin{aligned} f(0) &= 1 \\ f(2) &= 2^2 - 2 \cdot 2 + 1 \\ &= 1 \\ \Rightarrow f(0) &= f(2) = 1. \end{aligned} \qquad (2)$$

Then from (1) and (2), it is confirmed that Rolle's theorem can be applied. According to Rolle's theorem, there exists a point where $f'(x) = 0$ in the interval $(0, 2)$. □

Expansion of function

TAYLOR'S THEOREM

We now look at a result which allows us to compute the values of elementary functions like *sin*, *exp* and *log*. This theorem can be used to approximate these functions by polynomials (which are easy to compute) and provides an estimate of the error involved in the approximation.

Taylor's Theorem. *Let f be an $(n+1)$ times differentiable function on an open interval containing the points a and x. Then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

The function T_n defined by

$$T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n \quad \text{where } a_r = \frac{f^{(r)}(a)}{r!},$$

is called the *Taylor polynomial* of degree n of f at a. This can be thought of as a polynomial which approximates the function f in some interval containing a. The error in the approximation is given by the remainder term $R_n(x)$. If we can show $R_n(x) \to 0$ as $n \to \infty$ then we get a sequence of better and better approximations to f leading to a power series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which is known as the *Taylor series* for f. In general this series will converge only for certain values of x determined by the radius of convergence of the power series (see Note 17). When the Taylor polynomials converge rapidly enough, they can be used to compute approximate values of the function.

Connection with Mean Value Theorem.

When $n = 0$, Taylor's theorem reduces to the Mean Value Theorem which is itself a consequence of Rolle's theorem. A similar approach can be used to prove Taylor's theorem.

Proof of Taylor's Theorem.
The remainder term is given by

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Fix x and a. For t between x and a set

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n,$$

so that $F(a) = R_n(x)$. Then

$$\begin{aligned} F'(t) &= -f'(t) - f''(t)(x-t) + f'(t) - \frac{f'''(t)}{2!}(x-t)^2 + f''(t)(x-t) \\ &\quad - \cdots - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \end{aligned}$$

2

Now defining

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a),$$

we have $G(a) = 0$ and $G(x) = F(x) = 0$. Applying Rolle's theorem to the function G shows that there is a c between a and x with $G'(c) = 0$. Now

$$\begin{aligned} 0 = G'(c) &= F'(c) + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a) \\ &= -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a). \end{aligned}$$

But $F(a) = R_n(x)$ and rearranging the last equation gives

$$R_n(x) = F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \qquad \square$$

A useful consequence of Taylor's theorem is the following generalization of the second derivative test:

Basic (and important) Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{R})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (x \in \mathbb{R})$$

$$(1+x)^a = 1 + ax + \frac{a(a-1)x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \binom{a}{n} x^n \qquad (|x| < 1)$$

$$\text{where } \tbinom{a}{n} = a(a-1)\ldots(a-(n-1))/n! \qquad (a \in \mathbb{R})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \qquad (|x| < 1)$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad (|x| < 1)$$

Example. The Maclaurin series expansion of the exponential function is easy to find. If $f(x) = e^x$ then $f^{(n)}(x) = e^x$, so every $f^{(n)}(0)$ is 1, and

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find the values of x for which this is valid, we need to consider the remainder term (or use the Ratio Test alone; Note 17)

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \frac{x^{n+1}}{(n+1)!} e^c$$

for some c between 0 and x. It follows from the Ratio Test that the series $\sum(x^n/n!)$ converges for any x and hence the sequence $(x^n/n!)$ converges to 0. Therefore $R_n(x) \to 0$ for every $x \in \mathbb{R}$, so the Maclaurin series expansion is valid for every x.

Example. To find the Maclaurin series of the *sine* function we need to find its derivative of order n.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

It follows that $f^{(n)}(0) = 0$ if n is even, and alternates as $1, -1, 1, -1, \ldots$ for $n = 1, 3, 5, 7, \ldots$. Hence the Maclaurin series expansion is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

or

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

To find values of x for which this is valid, we need to consider the remainder term (or use the Ratio Test) which is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Now for each n, $f^{(n+1)}(c)$ is given by $\pm \sin c$ or $\pm \cos c$. The values of the sine and cosine functions always lie between -1 and 1, so

$$\frac{-x^{n+1}}{(n+1)!} \leq R_n(x) \leq \frac{x^{n+1}}{(n+1)!},$$

and since $x^{(n+1)}/(n+1)! \to 0$, we get $R_n(x) \to 0$ by the Squeeze Rule. This shows that the Maclaurin series expansion is valid for all $x \in \mathbb{R}$.

ABSTRACT

Content *definition, proof of Taylor's Theorem, n$^{\text{th}}$ derivative test for stationary points, Maclaurin series, basic Maclaurin series*

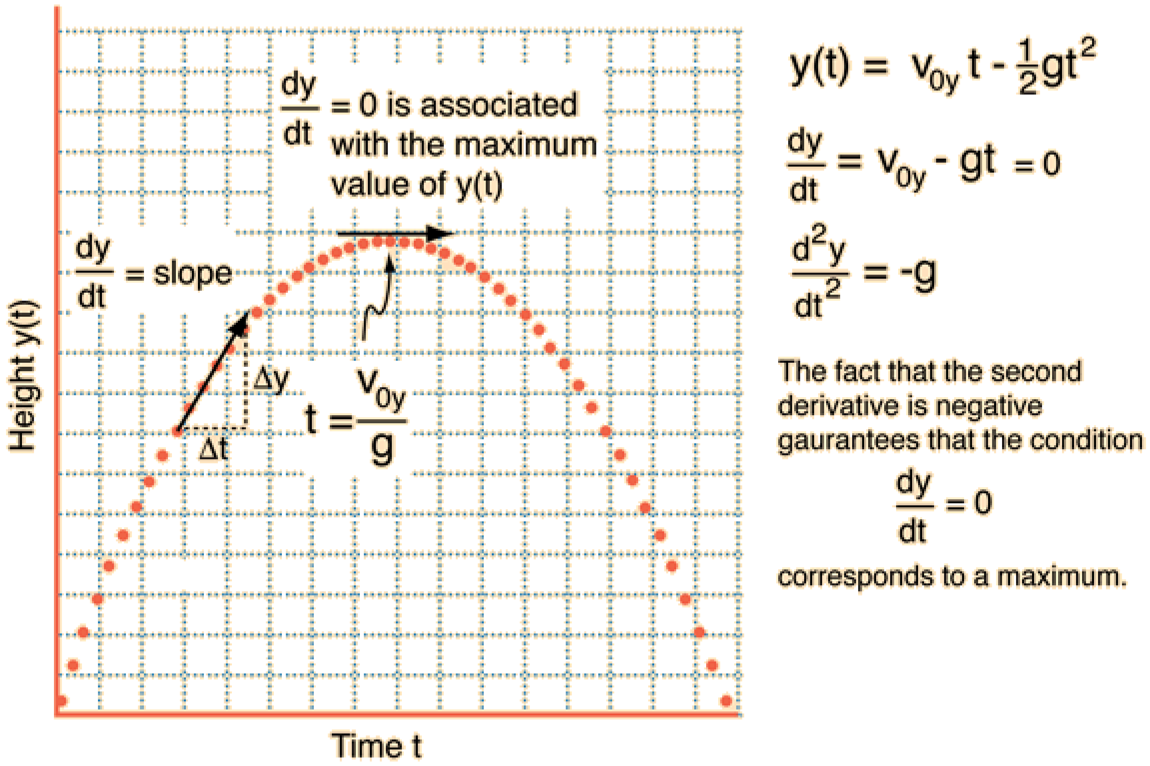
In this Note, we look at a Theorem which plays a key role in mathematical analysis and in many other areas such as numerical analysis. The well-known derivative test for maxima and minima of functions is generalised and Maclaurin Series are introduced.

Maxima and Minima from Calculus

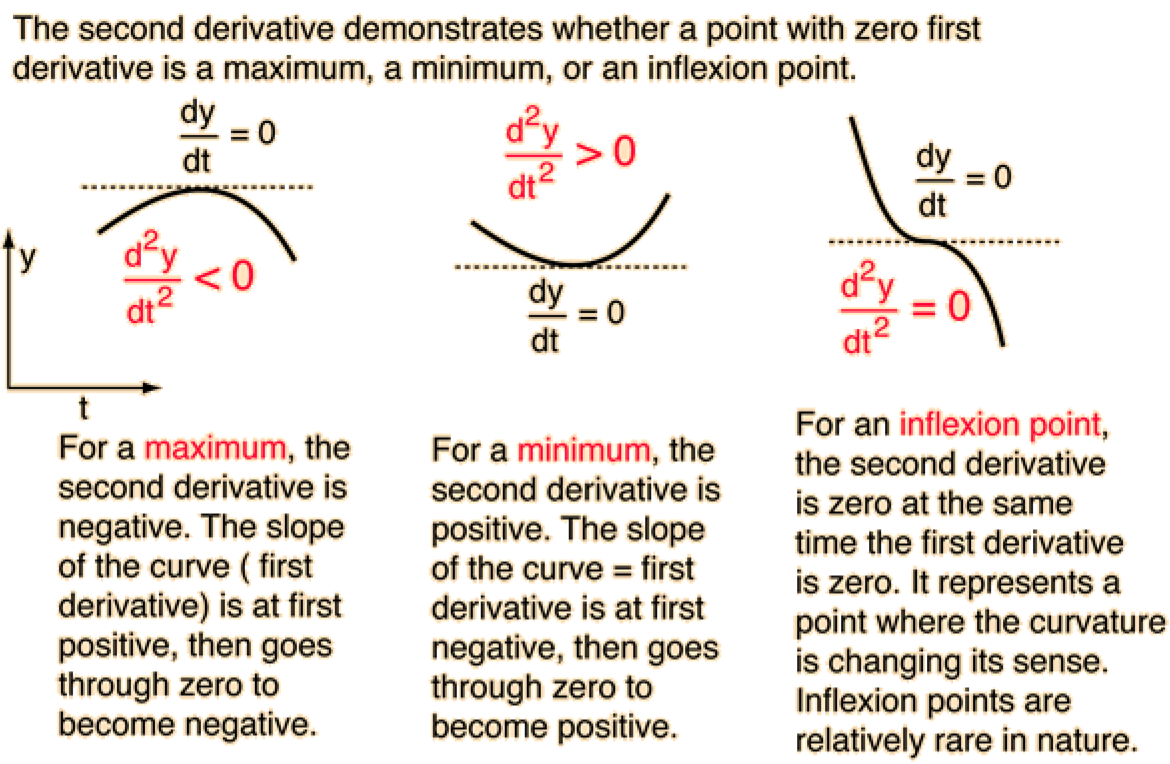
One of the great powers of calculus is in the determination of the maximum or minimum value of a function. Take f(x) to be a function of x. Then the value of x for which the derivative of f(x) with respect to x is equal to zero corresponds to a maximum, a minimum or an inflexion point of the function f(x).

example, the height of a projectile that is fired straight up is given by the motion equation

Taking $y_0 = 0$, a graph of the height y(t) is shown below



The derivative of a function can be geometrically interpreted as the slope of the curve of the mathematical function y(t) plotted as a function of t. The derivative is positive when a function is increasing toward a maximum, zero (horizontal) at the maximum, and negative just after the maximum. The second derivative is the rate of change of the derivative, and it is negative for the process described above since the first derivative (slope) is always getting smaller. The second derivative is always negative for a "hump" in the function, corresponding to a maximum.



Example: Find the maxima and minima for:

$y = 5x^3 + 2x^2 - 3x$

The derivative (slope) is:

$\frac{d}{dx}$

$y = 15x^2 + 4x - 3$

Which is quadratic with zeros at:

- $x = -3/5$
- $x = +1/3$

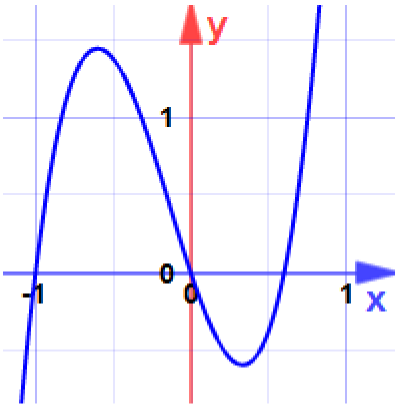
Could they be maxima or minima? (Don't look at the graph yet!)

The second derivative is **y″ = 30x + 4**

At x = −3/5: y″ = 30(−3/5) + 4 = −14 it is less than 0, so −3/5 is a local maximum

At x = +1/3: y″ = 30(+1/3) + 4 = +14it is greater than 0, so +1/3 is a local Minimum

Now you can look at the graph.)



Curve Sketching

The following steps are taken in the process of curve sketching:

1. Domain

Find the domain of the function and determine the points of discontinuity (if any).

2. Intercepts

Determine the x− and y−intercepts of the function, if possible. To find the x−intercept, we set $y=0$ and solve the equation for x. Similarly, we set $x=0$ to find the y−intercept. Find the intervals where the function has a constant sign ($f(x)>0$ and $f(x)<0$).

3. Symmetry

Determine whether the function is even, odd, or neither, and check the periodicity of the function. If $f(-x)=f(x)$ for all x in the domain, then f(x) is even and symmetric about the y−axis. If $f(-x)=-f(x)$ for all x in the domain, then f(x) is odd and symmetric about the origin.

4. Asymptotes

Find the vertical, horizontal and oblique (slant) asymptotes of the function.

5. Intervals of Increase and Decrease

Calculate the first derivative $f'(x)$ and find the critical points of the function. (Remember that critical points are the points where the first derivative is zero or does not exist.) Determine the intervals where the function is increasing and decreasing using the First Derivative Test.

6. Local Maximum and Minimum

Use the First or Second Derivative Test to classify the critical points as local maximum or local minimum. Calculate the y−values of the local extrema points.

7. Concavity/Convexity and Points of Inflection

Using the Second Derivative Test, find the points of inflection (at which $f''(x)=0$). Determine the intervals where the function is convex upward ($f''(x)<0$) and convex downward ($f''(x)>0$).

8 Graph of the Function

Sketch a graph of f(x) using all the information obtained above.

Example

$$f(x) = x^2 (x+3).$$

Solution.

The function is defined for all $x \in \mathbb{R}$. It has the following $x-$intercepts:

$$f(x) = 0, \quad \Rightarrow x^2 (x+3) = 0,$$
$$\Rightarrow x_1 = 0, x_2 = -3.$$

The $y-$intercept is equal to

$$f(0) = 0.$$

The function is positive on the intervals $(-3, 0)$ and $(0, +\infty)$ and negative on $(-\infty, -3)$.

The function is neither even nor odd, and it has no asymptotes.

Take the derivative:

$$f'(x) = \left(x^2(x+3)\right)' = \left(x^3 + 3x^2\right)'$$
$$= 3x^2 + 6x.$$

Find the critical points:

$$f'(x) = 0, \quad \Rightarrow 3x^2 + 6x = 0,$$
$$\Rightarrow 3x(x+2) = 0,$$
$$\Rightarrow x_1 = 0, x_2 = -2.$$

We can see from the sign chart that $x = -2$ is a point of maximum, and $x = 0$ is a point of minimum. The $y-$values of these points are

$$f\left(-2\right) = \left(-2\right)^2 \left(-2+3\right) = 4;$$

$$f\left(0\right) = 0.$$

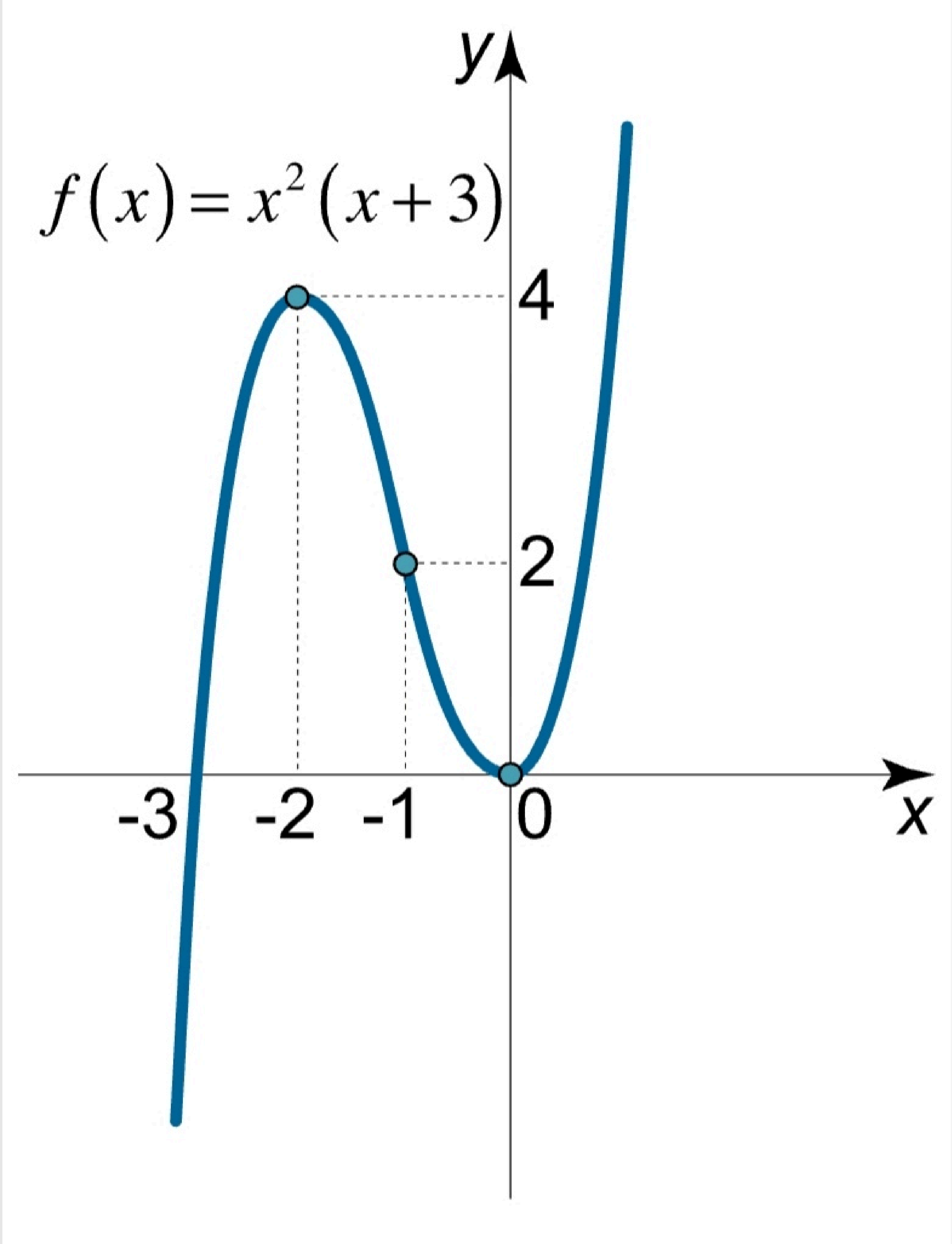
We differentiate once more to get the second derivative:

$$f''\left(x\right) = \left(3x^2 + 6x\right)' = 6x + 6.$$

$$f''\left(x\right) = 0, \quad \Rightarrow 6x + 6 = 0,$$
$$\Rightarrow x = -1.$$

The graph of the function is concave downward on $\left(-\infty, -1\right)$ and concave upward on $\left(-1, +\infty\right)$. Therefore, $x = -1$ is a point of inflection. The $y-$coordinate of this point is

$$f(-1) = (-1)^2(-1+3) = 2.$$



Successive differentiation & Liebnitz Theorem

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $\boldsymbol{f(x)}$ be a differentiable function and let its successive derivatives be denoted by $\boldsymbol{f'(x), f''(x), \ldots, f^{(n)}(x)}$.

Common notations of higher order Derivatives of $\boldsymbol{y = f(x)}$

1st Derivative: $\boldsymbol{f'(x)}$ **or** $\boldsymbol{y'}$ **or** $\boldsymbol{y_1}$ **or** $\boldsymbol{\frac{dy}{dx}}$ **or** $\boldsymbol{Dy}$

2nd Derivative: $\boldsymbol{f''(x)}$ **or** $\boldsymbol{y''}$ **or** $\boldsymbol{y_2}$ **or** $\boldsymbol{\frac{d^2y}{dx^2}}$ **or** $\boldsymbol{D^2y}$

$\vdots$

$\boldsymbol{n^{th}}$ Derivative: $\boldsymbol{f^{(n)}(x)}$ **or** $\boldsymbol{y^{(n)}}$ **or** $\boldsymbol{y_n}$ **or** $\boldsymbol{\frac{d^ny}{dx^n}}$ **or** $\boldsymbol{D^ny}$

1.2 Calculation of nth Derivatives

i. $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{e^{ax}}$

Let $y = e^{ax}$

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$\vdots$$
$$y_n = a^n e^{ax}$$

ii. $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{(ax+b)^m}$, $\boldsymbol{m}$ is a +ve integer greater than $\boldsymbol{n}$

Let $y = (ax+b)^m$

$$y_1 = m\,a(ax+b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$
$$\vdots$$
$$y_n = m(m-1)\ldots(m-n+1)a^n(ax+b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax + \text{b})^{m-n}$$

iii. n^{th} Derivative of $y = \log(ax+b)$

Let $y = \log(ax+b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\, a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$$

iv. n^{th} Derivative of $y = \sin(ax+b)$

Let $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax+b)$

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

v. n^{th} Derivative of $y = e^{ax}\sin(ax+b)$

Let $y = e^{ax}\sin(bx+c)$

$$y_1 = a\,e^{ax}\sin(bx+c) + e^{ax} b\cos(bx+c)$$

$$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}\,[\,r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)]$$

Putting $a = r\cos\alpha,\ b = r\sin\alpha$

$$= e^{ax}\ r\ \sin(bx+c+\alpha)$$

Similarly $y_2 = e^{ax}\ r^2\ \sin(bx+c+2\alpha)$

$$\vdots$$

$$y_n = e^{ax}\ r^n\ \sin(bx+c+n\alpha)$$

where $r^2 = a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly if $y = e^{ax}\cos(ax+b)$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha)$$

$$= e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	$\boldsymbol{n^{th}}$ **Derivative**
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & m>0, m>n \\ 0, & m>0, \quad m<n, \\ n!\, a^n, & m=n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, & m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$
$y = e^{ax} \sin(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
$y = e^{ax} \cos(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3\,(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow\ y_n = (-1)^{n+1}\, n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th}derivatives exist, then the n^{th}derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th}derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1} \frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n]$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

Example 12 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\, 25^{\frac{n}{2}} \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$

$$= e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x} \sin 4x\right)_n = x^2 e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) +$$

$$2nx e^{3x}\, 5^{n-1} \sin\left(4x + (n-1) \tan^{-1} \frac{4}{3}\right) +$$

$$n(n-1) e^{3x}\, 5^{n-2} \sin\left(4x + (n-2) \tan^{-1} \frac{4}{3}\right) + 0$$

$$= e^{3x}5^n \left[x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

Example 13 If $y = a\cos(\log x) + b\, sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\, cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\ sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{a\cos(logx) + b\ sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + (y_{n+1}x + n_{C_1}y_n.1) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14 If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

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Unit-4:Integration

List of Integral Formulas
The list of integral formulas are

∫ 1 dx = x + C
∫ a dx = ax+ C
∫ xn dx = ((xn+1)/(n+1))+C ; n≠1
∫ sin x dx = – cos x + C
∫ cos x dx = sin x + C
∫ sec2 dx = tan x + C
∫ csc2 dx = -cot x + C
∫ sec x (tan x) dx = sec x + C
∫ csc x (cot x) dx = – csc x + C
∫ (1/x) dx = ln |x| + C
∫ ex dx = ex+ C
∫ ax dx = (ax/ln a) + C ; a>0, a≠1
These integral formulas are equally important as differentiation formulas.

Some other important integration formulas are

- $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
- $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$
- $\int \sin^n(x)dx = \frac{-1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$
- $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$
- $\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)dx$
- $\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$
- $\int \csc^n(x)dx = \frac{-1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)dx$

Derivation of Integration By Parts Formula

If u(x) and v(x) are any two differentiable functions of a single variable y. Then, by the product rule of differentiation, we get;

$$\frac{d}{dx}(u(x)v(x)) = v(x)\frac{d}{dx}(u(x)) + u(x)\frac{d}{dx}(v(x))$$

Integrating both sides with respect to x,

$$\int \frac{d}{dx}(u(x)v(x))dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

then applying the definition of indefinite integral,

$$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Gives the formula for integration by parts. Since du and dv are differentials of a function of one variable x,

$$du = u'(x)dx \;\; dv = v'(x)dx$$

$$\int u(x)dv = u(x)v(x) - \int v(x)du$$

u′ is the derivative of u and v′ is the derivative of v.

To find the value of ∫vu′dx, we need to find the antiderivative of v′, present in the original integral ∫uv′dx.

Example 1: Find $\int x \cos x \, dx$

Solution: Let, The first function = f(x) = x and the second function = g(x) = cos x. Therefore, according to integration by parts, we have

$\int x \cos x \, dx = x \int \cos x \, dx - \int [(dx/dt) \int \cos x \, dx] \, dx$
$= x \sin x - \int \sin x \, dx$
$= x \sin x + \cos x + C.$

Let's try the other way round. Let, the first function = f(x) = cos x and the second function = g(x) = x. Therefore,
$\int x \cos x \, dx = \cos x \int x \, dx - \int \{[d(\cos x)/ dx] \int x \} \, dx$
$= (\cos x) (x^2/2) + \int (\sin x) (x^2/2) \, dx$

This is a really complex integral having a higher power of x. Hence, it is necessary to select the first and second function properly.

Example 2: Find ∫ log x dx

Solution: Do you know any function whose derivative is log x? Guessing it is difficult. Hence, let's take the first function f(x) = log x. So, the second function g(x) = 1. And, we know that ∫ 1 dx = x. Therefore, ∫ g(x) dx = x. Therefore, we have,

$\int (\log x.1)\, dx = \log x \int 1\, dx - \int \{[d(\log x)/dx] \int 1\, dx\}\, dx$

$= x \log x - \int (1/x)\, x\, dx = x \log x - \int 1\, dx$

$= x \log x - x + C$

Example 3: Find $\int x\, e^x\, dx$

Solution: We know that, $\int e^x\, dx = e^x$. Hence, we take the first function = f(x) = x and the second function = g(x) = e^x. Therefore, we have,

$\int x\, e^x\, dx = x \int e^x\, dx - \int [(dx/dx) \int e^x\, dx]\, dx = x\, e^x - \int 1.e^x\, dx$

$= xe^x - e^x + C$

Note:

- Integration by parts is not applicable for functions such as $\int \sqrt{x} \sin x \, dx$.
- We do not add any constant while finding the integral of the second function.
- Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, in cases where another function is an inverse trigonometric function or logarithmic function, then we take them as first function.

Ilate Rule

In integration by parts, we have learned when the product of two functions are given to us then we apply the required formula. The integral of the two functions are taken, by considering the left term as first function and second term as the second function. This method is called **Ilate rule.**

Suppose, we have to integrate $x\, e^{x}$, then we consider x as first function and e^{x} as the second function. So basically, the first function is chosen in such a way that the derivative of the function could be easily integrated. Usually, the preference order of this rule is based on some functions such as Inverse, Algebraic, Logarithm, Trigonometric, Exponent.

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) shows that differentiation and integration are inverse processes.

Part 1 (FTC1)

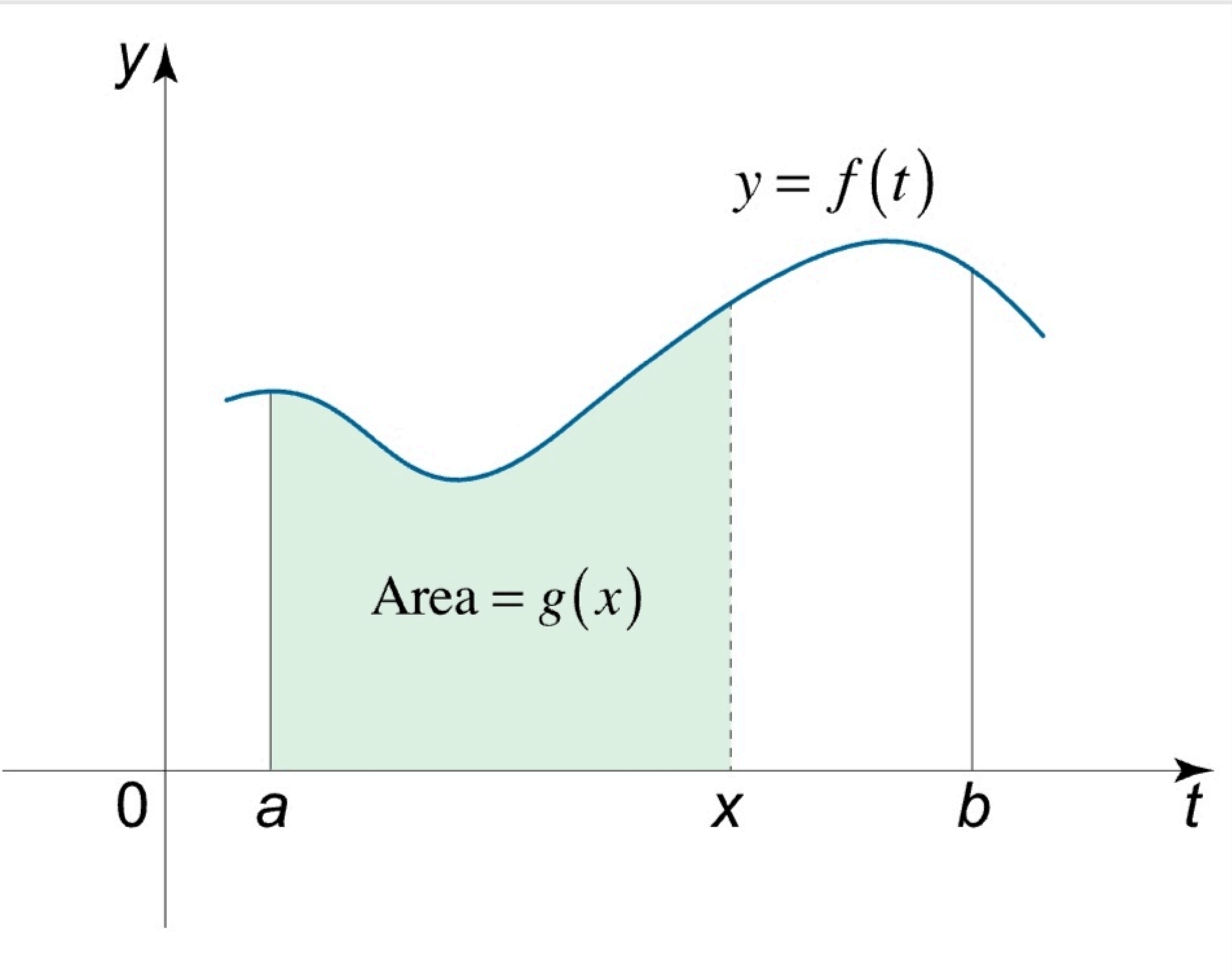
If f is a continuous function on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \;\; a \le x \le b$$

is an antiderivative of f, that is

$$g'(x) = f(x) \quad \text{or} \quad \frac{d}{dx}\left(\int_a^x f(t)\,dt\right)$$
$$= f(x) \,.$$

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x.



y
y = f(t)
Area = g(x)
0
a
x
b
t

Part 2 (FTC2)

The second part of the fundamental theorem tells us how we can calculate a definite integral.

If f is a continuous function on $[a, b]$ and F is an antiderivative of f, that is $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

or $\int_a^b F'(x)dx = F(b) - F(a)$.

To evaluate the definite integral of a function f from a to b, we just need to find its antiderivative F and compute the difference between the values of the antiderivative at b and a.

So the second part of the fundamental theorem says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function, but in the form F(b)−F(a).

Thus, the two parts of the fundamental theorem of calculus say that differentiation and integration are inverse processes.

This is defined as the definite integral as the limit of a sum.

Properties of Definite Integral

There are some properties of definite integral which could help to evaluate the problems based on it, easily.

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,d(t)$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$
- $\int_0^a f(x)\,dx = f(a - x)\,dx$

indefinite integral mean?

Answer: An indefinite integral refers to a function which takes the anti-derivative of another function. We visually represent it as an integral symbol, a function, and after that a dx at the end.

Formulae for Indefinite Integrals

Now that we already taken care of the concept of Integration, let’s take a quick look at some of the basic indefinite integrals formulae –

$$\int x^n dx = \frac{x^{n+1}}{(n+1)} + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int e^x dx = e^x + c$$

The Substitution Method

According to the substitution method, a given integral ∫ f(x) dx can be transformed into another form by changing the independent variable x to t. This is done by substituting x = g (t).

Consider, $I = \int f(x)\, dx$
Now, substitute $x = g(t)$ so that, $dx/dt = g'(t)$ or $dx = g'(t)dt$.
Therefore, $I = \int f(x)\, dx = \int f[g(t)]\, g'(t)dt$

It is important to note here that you should make the substitution for a function whose derivative also occurs in the integrand as shown in the following examples.

Example 1

Integrate sin(mx) with respect to x.

Solution: We know that the derivative of mx = m. Hence, let's substitute mx = t, so that mdx = dt. Therefore,

$\int \sin mx\, dx = 1/m \int \sin t\, dt$
$= -1/m \cos t + C$
$= -1/m \cos mx + C$

Example 2

Integrate 2x sin $(x^2 + 1)$ with respect to x.

Solution: We know that the derivative of $(x^2 + 1) = 2x$. Hence, let's substitute $(x^2 + 1) = t$, so that 2x dx = dt. Therefore,

$\int 2x \sin (x^2 + 1)\, dx = \int \sin t\, dt$
$= -\cos t + C$

$= -\cos t + C$

$= -\cos(x^2 + 1) + C$

Substitution Method for Some Important Integrals of Trigonometric Functions

- **$\int \tan x \, dx = \log |\sec x| + C$**

We know that tan x = sin x / cos x. Therefore,

$\int \tan x \, dx = \int (\sin x / \cos x) \, dx$.

Now, let's substitute cos x = t, so that sin x dx = – dt.

Therefore,

$\int \tan x \, dx = -\int (dt / t) = -\log |\cos x| + C$

Or, $\int \tan x \, dx = \log |\sec x| + C$

- **$\int \cot x \, dx = \log |\sin x| + C$**

We know that cot x = cos x / sin x. Therefore,

$\int \cot x \, dx = \int (\cos x / \sin x) \, dx$.

Now, let's substitute sin x = t, so that cos x dx = – dt.

Therefore,

$\int \cot x \, dx = \int (dt / t) = \log |t| + C = \log |\sin x| + C$

Different Forms Integration by Partial Fractions

Let's say that we want to evaluate ∫ [P(x)/Q(x)] dx, where P(x)/Q(x) is a proper rational fraction. In such cases, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition. Post this, integration can be carried out easily. The following image indicates some simple partial fractions which can be associated with various rational functions:

integration by partial fractions

Please note that A, B, and C are real numbers and their values should be determined suitably.

Question 1: Find ∫ dx / [(x + 1) (x + 2)]

Answer : The integrand is a proper rational function. Therefore, by using the form of partial fraction from the image above, we have:

1 / [(x + 1) (x + 2)] = A / (x + 1) + B / (x + 2) … (1)

Solving this equation, we get,

A (x + 2) + B (x + 1) = 1

Or, Ax + 2A + Bx + B = 1

x (A + B) + (2A + B) = 1

For LHS to be equal to RHS, we have

A + B = 0 and 2A + B = 1. On solving these two equations, we get

A = 1 and B = – 1.

Therefore, we have

1 / [(x + 1) (x + 2)] = 1 / (x + 1) – 1 / (x + 2)

Hence, ∫ dx / [(x + 1) (x + 2)] = ∫ dx / (x + 1) – ∫ dx /

$$(x + 2)$$

$$= \log|x + 1| - \log|x + 2| + C$$

Note: Equation (1) is true for all permissible values of x. Some authors use the symbol '≡' to indicate that the statement is an identity and use the symbol '=' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x.

Question 2: Find $\int [(x^2 + 1) / (x^2 - 5x + 6)]\, dx$

Answer : In this case, the integrand is NOT a proper rational function. Hence we divide $(x^2 + 1)$ by $(x^2 - 5x + 6)$ and get,

$(x^2 + 1) / (x^2 - 5x + 6) = 1 + (5x - 5) / (x^2 - 5x + 6)$

$= 1 + (5x - 5) / (x - 2)(x - 3)$

Now, let's look at the second half of the above equation and let

$(5x - 5) / (x - 2)(x - 3) = A / (x - 2) + B / (x - 3)$

On solving it, we get

$5x - 5 = A(x - 3) + B(x - 2) = Ax - 3A + Bx - 2B = x(A + B) - (3A + 2B)$

Comparing the coefficients of the x term and constants, we get

$A + B = 5$ and $3A + 2B = 5$. Further, on solving these two equations, we get

$A = -5$ and $B = 10$.

Hence, we have

$(x^2 + 1) / (x^2 - 5x + 6) = 1 - 5 / (x - 2) + 10 / (x - 3)$

Therefore, $\int [(x^2 + 1) / (x^2 - 5x + 6)]\, dx = \int dx - 5 \int 1 / (x - 2) + 10 \int 1 / (x - 3)$

$= x - 5\log |x - 2| + 10\log |x - 3| + C$

3.1 INTRODUCTION

In the first two units of this block we have introduced the concept of a definite integral and have obtained the values of integrals of some standard forms. We have also studied two important methods of evaluating integrals, namely, the method of substitution and the method of integration by parts. In the solution of many physical or engineering problems, we have to integrate some integrands involving powers or products of trigonometric functions. In this unit we shall devise a quicker method for evaluating these integrals.We shall consider some standard forms of integrands one by one, and derive formulas to integrate them.

The integrands which we will discuss here have one thing in common. They depend upon an integer parameter. By using the method of integration by parts we shall try to express such an integral in terms of another similar integral with a lower value of the parameter. You will see that by the repeated use of this technique, we shall be able to evaluate the given integral.

Objectives

After reading this unit you should be able to derive and apply the reduction formulas for

- $\int x^n e^x dx$
- $\int \sin^n x dx$, $\int \cos^n x\, dx$ $\int \tan^n x dx$, etc.
- $\int \sin^m x \cos^n x dx$
- $\int e^{ax} \sin^n x\, dx$
- $\int \sinh^n x dx$, $\int \cosh^n x dx$

3.2 REDUCTION FORMULA

Sometimes the integrand is not only a function of the independent variable, but also depends upon a number n (usually an integer). For example, in $\int \sin^n x\, dx$, the integrand $\sin^n x$ depends on x and n. Similarly, in $\int e^x \cos mx\, dx$, the integrand $e^x \cos mx$ depends on x and m. The numbers n and m in these two examples are called parameters. We shall discuss only integer parameter here.

3.3.1 Reduction Formulas for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$

In this sub-section we will consider integrands which are powers of either sinx or cosx. Let's take a power of sinx first. For evaluating $\int \sin^n x dx$, we write

$$I^n = \int \sin^n x dx = \int \sin^{n-1} x \sin x dx, \text{ if } n > 1.$$

Taking $\sin^{n-1}x$ as the first function and sin x as the second and integrating by parts, we get

$$I_n = -\sin^{n-1} x \cos x - (n-1) \int \sin^{n-2} x \cos x (-\cos x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) [\int \sin^{n-2} x (1-\sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) [\int \sin^{n-2} x dx - \int \sin^n x) dx]$$

$$= -\sin^{n-1} x \cos x + (n-1) [I_{n-2} - I_n]$$

Hence,

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

That is, $nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$ Or

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the reduction formula for $\int \sin^n x dx$ (valid for $n \geq 2$).

for $n = 1$

$\int \sin^n x dx = \int \sin x \, dx$

$-\cos x + c$

Example 2 We will now use the reduction formula for $\int \sin^n x dx$ to evaluate the definite integral, $\int_0^{\pi/2} \sin^5 x dx$. We first observe that

$$\int_0^{\pi/2} \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} \Bigg]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx$$

$$= \frac{n-1}{n} \int^{\pi/2} \sin^{n-2} x dx, \; n \geq 2.$$

Thus, $\int_0^{\pi/2} \sin^5 x dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x dx$

$$= \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x dx$$

$$= \frac{8}{15} (-\cos x) \Bigg]_0^{\pi/2}$$

$$= \frac{8}{15}$$

Let us now derive the reduction formula for $\int \cos^n x dx$. Again let us write

$$I_n = \int \cos^n x dx = \int \cos^{n-1} x \cos x dx, \; n > 1.$$

Integrating this integral by parts we get

$$I_n = \int \cos^{n-1} x \sin x - \int (n-1) \cos^{n-2} x (-\sin x). \sin x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1-\cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) (I_{n-2} - I_n)$$

By rearranging the terms we get

$$I_n = \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

This formula is valid for $n \geq 2$. What happens when $n = 0$ or 1? You will agree that the integral in each case is easy to evaluate.

As we have observed in Example 2,

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx,\ n \geq 2.$$

Using this formula repreatedly we get

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x dx, \text{ if } n \text{ is an odd number, } n \geq 3. \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} dx, \text{ if } n \text{ is an even number, } n \geq 2. \end{cases}$$

This means

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \text{ if } n \text{ is odd, and } n \geq 3 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2}, \text{ if } n \text{ is even } n \geq 2 \end{cases}$$

We can reverse the order of the factors, and write this as

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \cdots\cdots \frac{n-3}{n-2} \cdot \frac{n-1}{n}. \text{ if } n \text{ is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} \cdot \frac{\pi}{2}. \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

Arguing similarly for $\int_0^{\pi/2} \cos^n x dx$ we get

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{2}{3} \cdot \frac{4}{5} \cdots\cdots \frac{n-1}{n}. \text{if } n \text{ is odd, and } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} \cdot \frac{\pi}{2} \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

We are leaving the proof of this formula to you as an exercise See E1)

E E1) Prove that

$$\int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{2}{3} \cdot \frac{4}{5} \cdots\cdots \frac{n-1}{n}, \text{ if } n \text{ is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} \cdot\cdot \frac{\pi}{2}, \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

Gamma and Beta functions

Definition.
For x positive we define the **Gamma function** by

This integral cannot be easily evaluated in general, therefore we first look at the Gamma function at two important points. We start with $x = 1$:

Now we look at the value at $x = 1/2$:

The last integral cannot be evaluated using antiderivative . However, this particular definite integral is very important (for instance in probability), so people eventually found a trick to find its value.

To find the value of the Gamma function at other points we deduce an interesting identity using integration by part:

The limit is evaluated using l'Hospital's rule several times. We see that for x positive we have

If we apply this to a positive integer n, we get

So we see that the Gamma function is a generalization of the factorial function. It is possible to show that the limit of the Gamma function at 0 from the right is infinity, the graph looks like this:

Since at integer points, the value of the Gamma function is given by the factorial, it follows that the Gamma function grows to infinity even faster than exponentials.

Definition.
For x,y positive we define the **Beta function** by

Using the substitution $u = 1 - t$ it is easy to see that

To evaluate the Beta function we usually use the Gamma function. To find their relationship, one has to do a rather complicated calculation involving change of variables (from rectangular into tricky polar) in a double integral. This is beyond the scope of this section, but we include the calculation for the sake of completeness:

Thus

B.C.A study

Unit-5 :Vector alegbra

Introduction to Vectors

A->B

DEFINITION: Vectors were developed to provide a compact way of dealing with multidimensional situations without writing every bit of information. Vectors are quantities that have magnitude and direction, they can be denoted in three ways: in bold (**r**), underlined (r) or .

The position point of a vector is defined using Cartesian co-ordinates: it uses the coordinates of the OX, OY and OZ axes where O is the origin. We will be looking at vectors in 3 dimensional space in Cartesian coordinates. Similar ideas hold for vectors in n dimensional space (n-vectors).

graph1

Addition and Subtraction of vectors

DEFINITION: We will look at two laws involving addition and subtraction; The commutative law and the Associative law.

The commutative law:

Addition and subtraction of vectors obeys the commutative law,

matixA

matixA

This means that:

matixA

In terms of the following components

components1

For addition:

components2

For subtraction:

components3

The Associative law:

This uses different routes to get to the same final desination

graph2

graph2

In terms of the following components:

components4

components5

The length and unit vector of a vector

The length of a vector :

DEFINITION: For any vector like this one

components

The length of a (which is denoted by |a|) is given by:

a

The unit vector of a vector

DEFINITION: A unit vector is a vector with unit length 1. By standard convention we let i, j and k be unit vectors along the positive x, y and z axes, so in terms of components:

a

a

Scalar multiplication of vectors

DEFINITION: This involves the multiplying of a vector by a scalar, i.e. a number.

In terms of the following components:

components

components

Scalar multiplication will change the length of the vector and if the factor λ is negative the vector will point to the opposite direction. Scalar multiplication satisfies the following properties where a and b are vectors, λ and μ and are scalars.

components

Scalar product

Let us take two vectors.

Vectors

Then the scalar product of a and b, denoted by a.b is

scalar product

Setting these equal to each other gives

vector product

The angle between two vectors

Using the two above formulae and setting them equal to each other as shown below we are able to calculate the angle between two vectors.

a

Vector Product

DEFINITION: The vector product is fundamentally different from the scalar product. The vector product of two vectors is a vector but the scalar product is a scalar. The vector product is given by:

a

where
|a| is the length of a
θ is the angle between vectors
n is the unit vector perpendicular to a and b whose direction is determined by the left hand skew rule.

a

For vectors a × b is found by using the following:

a

For simplicity this can be written in terms of determinants.

a

Scalar Triple Product

By the name itself, it is evident that scalar triple product of vectors means the product of three vectors. It means taking the dot product of one of the vectors with the cross product of the remaining two. It is denoted as

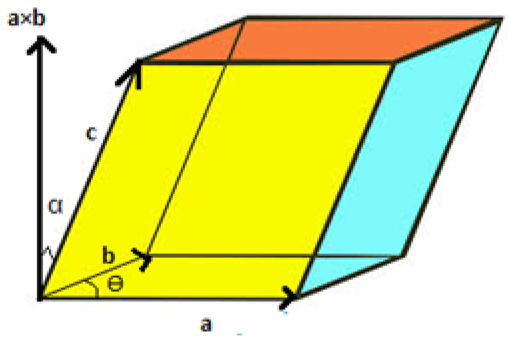
[a b c] = (a × b) . c

The following conclusions can be drawn, by looking into the above formula:

i) The resultant is always a scalar quantity.

ii) Cross product of the vectors is calculated first followed by the dot product which gives the scalar triple product.

iii) Talking about the physical significance of scalar triple product formula it represents the volume of the parallelepiped whose three co-terminous edges represent the three vectors a,b and c. The following figure will make this point more clear.



According to this figure, the three vectors are represented by the coterminous edges as shown. The cross product of vectors **a** and **b** gives the area of the base and also the direction of the cross product of vectors is perpendicular to both the vectors.As volume is the product of area and height, the height in this case is given by the component of vector **c** along the direction of cross product of **a** and **b** **.** The component is given by c cos α .

Thus, we can conclude that for a Parallelepiped, if the coterminous edges are denoted by three vectors and **a,b** and **c** then,

Volume of parallelepiped = **(a × b)** c cos α = **(a × b) . c**

Where α is the angle between **(a × b)** and**.c**

We are familiar with the expansion of cross product of vectors. Keeping that in mind, if it is given that **a** = a1i^+a2j^+a3k^, **b =** b1i^+b2j^+b3k^ , and **c =** c1i^+c2j^+c3k^ then,we can express the above equation as,

(a × b) . c = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$. ($c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$)

This indicates the dot product of two vectors. Using properties of determinants, we can expand the above equation as,

(a × b) . c =

$$\begin{vmatrix} \hat{i}.(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) & \hat{j}.(c_1\hat{i} + c_2\hat{j} + \\ a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

According to the dot product of vector properties,

$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k}$= 1 (As cos 0 = 1)

⇒ $\hat{i}.(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$ = c_1

⇒ $\hat{j}.(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$ = c_2

⇒ $\hat{k}.(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$ = c_3

⇒ **(a × b) . c =**

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

[a b c] = $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Properties of Scalar Triple Product:

i) If the vectors are cyclically permuted,then

(a × b) . c = a.(b × c)

ii) The product is cyclic in nature, i.e,

[a b c] = [b c a] = [c a b] = – [b a c] = – [c b a] = – [a c b]

Example: Three vectors are given by,**a =** $\hat{i} - \hat{j} + \hat{k}$ **, b =** $2\hat{i} + \hat{j} + \hat{k}$ **,and c =** $\hat{i} + \hat{j} - 2\hat{k}$.

By using the scalar triple product of vectors, verify that **[a b c] = [b c a] = – [a c b]**

Solution: First of all let us find **[a b c]**.

[a b c] = (a × b) . c

We know **[a b c] =** $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\Rightarrow [\mathbf{a\ b\ c}] = \begin{vmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow [\mathbf{a\ b\ c}] = -7$$

Now let us evaluate **[b c a]** and **[a c b]** similarly,

$$\Rightarrow [\mathbf{b\ c\ a}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -7$$

$$\Rightarrow [\mathbf{a\ c\ b}] = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 7$$

Hence it can be seen that **[a b c] = [b c a] = − [a c b]**

Try to recall the properties of determinants since the concept of determinant helps in solving these types of problems easily.

iii) If the triple product of vectors is zero, then it can be inferred that the vectors are coplanar in nature.

The triple product indicates the volume of a parallelepiped. If it is zero, then such a case could only arise when any one of the three vectors is of zero magnitude. The direction of the cross product of **a** and **b** is perpendicular to the plane which contains **a** and **b**. The dot product of the resultant with **c** will only be zero if the vector c also lies in the same plane. This is because the angle between the resultant and **C** will be 90∘ and cos 90∘..

Thus, by the use of the scalar triple product, we can easily find out the volume of a given parallelepiped

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